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Mathematical Modeling the Electrical Impedance of the Piezoceramic Disk Oscillating in a Wide Frequency Range (Part 1. Low Frequencies)

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The article presents the results of mathematical modeling and analysis of the electrical impedance of a piezoceramic disk that undergoes oscillations at low frequencies, i.e., when the length of the elastic wave significantly (by an order of magnitude or more) exceeds the radial size of the disk. Thus, the proposed mathematical model of disk-shaped ceramic elements of piezoelectric transducers, which are an important component of modern communication devices, environmental sensors, precision equipment, medical devices, etc. A key characteristic of the mathematical model described in the article is its ability to determine analytical dependencies that allow estimating such fundamental electrical properties of the piezoceramic disk element as electrical impedance and quasi-static electrical capacitance, thereby significantly simplifying the calculation of such an element already at the design stage. The static dielectric permittivity of a piezoceramic disk vibrating at low frequencies has been investigated. The calculated value of this parameter, based on the physical constants' characteristic of the piezoceramic of the PZT (lead zirconate titanate) type, is 1.844 times higher compared to the high-frequency (dynamic) dielectric permittivity. It has been found that in the low-frequency range, when the mechanical stresses in the vibrating piezoceramic disk approach zero and the direct piezoelectric effect is almost negligible, the electrical impedance of such a disk can be described as the reactive resistance of a capacitor with electrical capacitance equivalent to the quasi-stationary capacitance of the disk. This is confirmed by a high degree of convergence between theoretical data and experimental results, with discrepancies not exceeding 6%. The results obtained in the article can be valuable for scientific research in the fields of precision instrument engineering and radio equipment manufacturing. Additionally, they have practical applications in the development and production of high-tech equipment.

Keywords: piezoelectric transducer; acoustoelectronics; mathematical modeling; impedance; disk element

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Introduction

In modern instrument engineering and radio equipment manufacturing, special attention is given to the development and improvement of devices made from piezoelectric materials, among which disk-shaped piezoceramic elements hold a significant place [1]. They are an integral part of communication devices, environmental sensors, precision equipment, and medical devices [2]. Considering this fact, understanding the peculiarities of the operation of disk-shaped piezoceramic elements plays a crucial role in enhancing their effectiveness.

One of the key parameters determining the efficiency of using such piezo elements is their electrical impedance [3], especially under oscillations across a wide frequency range. Studying this issue holds significant theoretical and practical importance as it enables the optimization of the design and utilization of piezoceramic sensors in a broad spectrum of applications.

This article addresses the issue of mathematical modeling of the electrical impedance of a piezoceramic disk oscillating across a wide frequency range, with a particular focus on its performance at low frequencies. Such mathematical modeling allows for replicating the behavior of disk-shaped piezo elements under low-frequency conditions and identifying optimal operating modes. This is crucial for researchers and engineers involved in the development of relevant equipment. Furthermore, it contributes to a deeper understanding of the physical processes underlying the operation of disk-shaped elements made of piezoceramic materials.

1 The relevance of the research based on the results of the publications' analysis

Modern research in the field of mathematical modeling of functional elements from piezoceramic materials emphasizes the importance of a precise understanding of their electrical (including impedance) and mechanical properties. This encourages further improvement and development of an effective instrumentation base. An analysis of recent Ukrainian and international publications indicates that this topic is actively researched by the scientific community and many specialists specializing in piezotechnology and acoustic radio equipment [4].

Ukrainian scientists, including practicing researchers from the research institutes of the National Academy of Sciences of Ukraine and higher education institutions in Ukraine (notably Derkach O., Ishchuk V., Kirilyuk V., Kuzenko D., Petrishchev O., and others [5–7]), make a significant contribution to the development of mathematical models of piezoelectric materials. Their works often focus on practical applications and the impact of various factors on the properties of piezoceramics, making the modeling more accurate and efficient for real operating conditions.

International research, particularly from Europe and Asia, often focuses on improving materials and manufacturing technologies for piezoceramics, directly influencing their electrical characteristics, including impedance. Among such studies, works by Cao P., Chen J., Gogoi N., Kirchner J., Wang Z., Zhang S., Zhou K., and others [8–10] are noteworthy. Popular topics also include research on impedance changes at different frequencies of vibrations (notably the works of Han H., Cheng C., Xiong X-G., Nguyen T. T., Hoang N. D., and others [11,12]), allowing for the optimization of piezoelement utilization in various devices, from medical sensors to industrial sensors.

A significant number of works are dedicated to the development of new methodologies and software for modeling various processes occurring in disk elements made of piezoceramic materials. Among these, the contributions of scientists such as Fischer G., Brissaud M., Kenji U., and others [13, 14] are notable. The results obtained in their studies help to better understand and predict the behavior of piezoelements under different conditions, playing a crucial role in the development of advanced technologies and improving the accuracy of applying disk elements made of piezoceramic materials.

However, after analyzing scientific papers from open sources dedicated to mathematical modeling of electrical properties of piezoelectric transducers, the absence of a mathematical model for disk-shaped piezoelectric transducers that would allow for the precise and relatively straightforward determination of their electrical impedance has been identified.

Therefore, the topic of mathematical modeling of the electrical impedance of a piezoceramic disk in the low-frequency range is quite relevant and promising, considering the constant development of technologies and the increasing demand for high-precision devices in various industrial sectors.

2 Formulation and solution of the mathematical modeling task for a piezoceramic disk-shaped transducer

In Figure 1, we investigate a disk with thickness α that is many times less than its radius R. The surfaces of the disk $z = 0$ and $z = \alpha$ (z is the coordinate axis in the cylindrical coordinate system ρ, φ, z, the origin of which aligns with the center of the disk's lower surface) has electrode coating, namely, a thin layer of silver (below 10 μm) by technology as described in [15]. Electric potential $U_0 e^{i\omega t}$ is applied to the top surface $z = \alpha$ (U_0 is amplitude value of the electric potential selected from the condition $U_0/\alpha << 0,1E_0$, where $E_0 \cong 2$ MV/m is the electric field strength polarizing the disk material, which guarantees the absence of nonlinear effects; $i = \sqrt{-1}$ is an imaginary unit; ω is the angular frequency of electric potential sign inversion; t denotes time). The lower electrode surface $z = 0$ is grounded, i.e., the potential on the surface is zero.

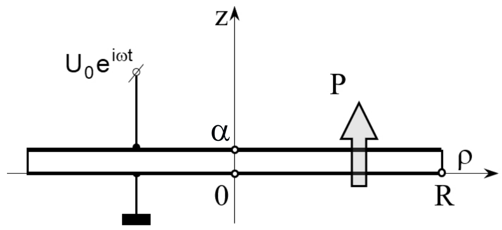

Fig. 1. Computational scheme for an oscillating piezoceramic disk

The electrical potential applied to the disk induces an electric field in the disk's volume, which displaces the ions of zirconium, titanium, lead, and oxygen from the equilibrium position. As a result of the harmonically time-varying disk deformation, polarization charges arise and interact with electric charges on the electrode disk surfaces, the electric charges being delivered to these surfaces by an electric potential difference generator. The resultant electric charge $Qe^{i\omega t}$, that is present on the surface $z = \alpha$, arouses electric current $Ie^{i\omega t}$ with its electric field in the conductor connecting the surface $z = \alpha$ to the electrical generator output. At any time given, the equation holds $Ie^{i\omega t} = -\partial Q/\partial t = -i\omega Qe^{i\omega t}$, i.e., the amplitudes of the electric current and charge on the $z = \alpha$ surface are related by a linear relationship $I = -i\omega Q$.

Obviously, the $Z_{el}(\omega)$ electrical impedance of the oscillating disk is subject to the Ohm's law for the circuit section [16], therefore

$$Z_{el}\left(\omega\right) = \frac{U_0}{I} = -\frac{U_0}{i\omega Q}. \quad (1)$$

The $\vec{D}\left(\rho,\, \varphi,\, z\right)$ amplitude value of the component of the electric induction vector component which is normal for this surface will determine the σ_0 surface density amplitude value of the electric charge. In the conditions under consideration $\sigma_0 = D_z(\rho, z)$ and, since all physical fields in the oscillating piezoceramic disk a priori have axial symmetry, we therefore have the equation

$$Q = \int\limits_S \sigma_0 dS = \int\limits_0^{2\pi} \int\limits_0^R \rho\, D_z\left(\rho, z\right)\, d\rho\, d\varphi = \\ = 2\pi \int\limits_0^R \rho\, D_z\left(\rho\,, z\right)\, d\rho. \quad (2)$$

The electrical state of the disk will be determined by the electric polarization law for the dielectric with piezoelectric properties [17], and, in terms of the amplitude values of the physical field characteristics harmoniously varying in time, the law reads

$$D_k = e_{knm}\varepsilon_{nm} + \chi^{\varepsilon}_{kj} E_j, \quad k,\, n,\, m,\, j = 1,\, 2,\, 3, \quad (3)$$

where D_k is the amplitude value of the k-th electric induction vector component (the unit for which is coulomb divided by square meter); e_{knm} is tensor component of piezoelectric modules (the unit is coulomb divided by square meter); ε_{nm} is amplitude value of the infinitesimal strain tensor component (dimensionless quantity); χ^{ε}_{kj} denotes components of the dielectric constant tensor, which can be determined experimentally in the mode of constant (equal to zero) elastic deformations (upper symbol ε); E_j is the amplitude value of the j-th component of the electric field strength vector in the volume of the deformed piezoelectric. The notation of relation (3) assumes by default that the convention on summation over twice repeated indices is fulfilled. There exists a one-to-one correspondence between the indices in the coordinate axes of the right-handed Cartesian coordinate system and the symbols on the axes in the cylindrical coordinate system, namely: $1 \leftrightarrow \rho$; $2 \leftrightarrow \varphi$ and $3 \leftrightarrow z$.

The components of the infinitesimal strain tensor satisfy the generalized Hooke's law for an elastic medium with piezoelectric properties [17] which has the following notation:

$$\sigma_{ij} = c^E_{ijk\ell}\varepsilon_{k\ell} - e_{kij} E_k, \quad (4)$$

where σ_{ij} is the amplitude value of the mechanical stress tensor component (the unit is newton divided by square meter or pascal); $c^E_{ijk\ell}$ is the elastic modulus tensor component, which is determined experimentally in the mode of constant (equal to zero) electric field strength (upper symbol E) within the volume of a strained piezoelectric.

Elastic stresses σ_{ij} and inertial forces that arise in the volume of a dynamically deformed solid body are related by Newton's second law in its differential form or, in other words, by the equations of motion, which, in the case of an axisymmetric stress-strain state varying in time according to the harmonic law, are noted in a cylindrical coordinate system as follows [18]:

$$\frac{\partial\, \sigma_{\rho\rho}}{\partial\, \rho} + \frac{\partial\, \sigma_{\rho z}}{\partial\, z} + \frac{1}{\rho}\left(\sigma_{\rho\rho} - \sigma_{\varphi\varphi}\right) + \rho_0\omega^2 u_\rho = 0, \quad (5)$$

$$\frac{1}{\rho}\frac{\partial}{\partial\, \rho}\left(\rho\sigma_{z\rho}\right) + \frac{\partial\, \sigma_{zz}}{\partial\, z} + \rho_0\omega^2 u_z = 0, \quad (6)$$

where ρ_0 is piezoceramic density; u_ρ and u_z are amplitude values of the displacement vector components of material particles, that is, infinitesimal volumes of piezoceramics. Normal and shear stresses on the disk surfaces must satisfy Newton's third law. If the disk oscillates in a vacuum or in the air – both situations are similar and mean that the disk does not have mechanical contacts with other material objects (Fig. 1) – the following conditions will be satisfied on the disk surfaces:

$$\left.\sigma_{z\rho}\right|_{z\,=\,0\,,\,\alpha} = \left.\sigma_{zz}\right|_{z\,=\,0\,,\,\alpha} = 0\,\forall\,\rho \in [0,\, R]\,, \quad (7)$$

$$\left.\sigma_{\rho z}\right|_{\rho\,=\,R} = \left.\sigma_{\rho\rho}\right|_{\rho\,=\,R} = 0\,\forall\, z \in [0,\, \alpha]\,. \quad (8)$$

Since the components of the displacement vector ($\varepsilon_{\rho\rho} = \partial u_\rho/\partial\rho$, $\varepsilon_{\varphi\varphi} = u_\rho/\rho$, $\varepsilon_{zz} = \partial\, u_z/\partial z$ and $\varepsilon_{\rho z} = (\partial u_\rho/\partial z + \partial u_z/\partial \rho;)/2$ in the problem under consideration) determine the deformations, we argue that a mathematical description of the electrical impedance that is adequate to the real situation presupposes an adequate mathematical description of the dynamic stress-strain state of the oscillating piezoceramic disk. Naturally, the qualitative and quantitative characteristics of the stress-strain state within the disk volume are significantly influenced by the electric field, which is the algebraic sum of the electric field created by the electric potential difference generator (hereinafter referred to as the electric field of an external source), and the electric field that arises from the displacement of ions from the equilibrium position (direct piezoelectric effect). Further, this field will be referred to as the internal electric field. The intensity vector $\vec{E}\left(\rho, \varphi, z\right) e^{i\omega t}$ of the total electric field or, as previously mentioned, the electric field within the volume of a deformable piezoelectric, satisfies Maxwell's equations, which in terms of the amplitude values of the harmonically time-varying physical fields may be written as:

$$rot\, \vec{H} = \vec{J} + i\omega\vec{D}, \quad (9)$$

$$rot\, \vec{E} = -\, i\omega\vec{B}, \quad (10)$$

where $\vec{H}$ and $\vec{B}$ are amplitude values of the intensity and induction vectors of the alternating magnetic field, while $\vec{B} = \mu_0 \vec{H}$ ($\mu_0 = 4\pi \cdot 10^{-7}$ H/m is the magnetic permeability of vacuum, i.e., the magnetic constant); $\vec{J} = r\vec{E}$ is surface conduction current density; r is the specific electrical conductivity of the medium. Piezoelectric ceramics are good electric current insulators. Without excessive idealization, we assume $r = 0$, equation (9) is therefore

$$rot\,\vec{H} = i\omega\vec{D}. \tag{11}$$

Calculating the divergence from the left and right sides of equation (11), we conclude that

$$div\,\vec{D} = 0. \tag{12}$$

Condition (12) determines the analytical properties of the electric induction vector and is usually considered the condition of absent free electricity carriers in the volume of the deformable piezo-dielectric.

The work [19] indicates that, within the frequency range of the order of several megahertz, the alternating magnetic field in the volume of deformable piezoceramics is insignificant enough to consider that $rot\,\vec{E} \approx 0$. From the last equality it follows that the electric field in the volume of deformable piezoceramics is irrotational, i.e., potential, and can be described with a scalar electric potential $\varphi(\rho, \varphi, z)\, e^{i\omega t}$. In this case, the amplitude value of the electric field strength vector will be determined through the scalar potential amplitude value by the standard method:

$$\vec{E} = -grad\,\varphi. \tag{13}$$

By substituting definition (13) into relations (3), and the obtained results into condition (12), we obtain a second-order partial differential equation, which we can rewrite in a cylindrical coordinate system as

$$\begin{gathered}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\left(e_{1nm}\varepsilon_{nm} - \frac{\chi^{\varepsilon}_{1j}}{h_j}\frac{\partial\varphi}{\partial q_j}\right)\right] + \\ + \frac{1}{\rho}\frac{\partial}{\partial\varphi}\left(e_{2nm}\varepsilon_{nm} - \frac{\chi^{\varepsilon}_{2j}}{h_j}\frac{\partial\varphi}{\partial q_j}\right) + \\ + \frac{\partial}{\partial z}\left(e_{3nm}\varepsilon_{nm} - \frac{\chi^{\varepsilon}_{3j}}{h_j}\frac{\partial\varphi}{\partial q_j}\right) = 0,\end{gathered} \tag{14}$$

where h_j are Lamé coefficients of a cylindrical coordinate system ($h_1 = 1$; $h_2 = \rho$; $h_3 = 1$); q_j ($q_1 = \rho$; $q_2 = \varphi$ и $q_3 = z$) is the j-th coordinate in the cylindrical coordinate system. The solution to this equation for the object shown in Fig. 1, i.e., the $\varphi(\rho, z)$ scalar potential must satisfy obvious conditions:

$$\varphi(\rho, z)|_{z=0} = 0, \quad \varphi(\rho, z)|_{z=\alpha} = U_0. \tag{15}$$

The approximate condition must be satisfied on the lateral surface of the disk $\rho = R$ [19]

$$\left.\frac{\partial\varphi(\rho, z)}{\partial\rho}\right|_{\rho=R} \cong 0. \tag{16}$$

Thus, providing the analytical description for $Z_{el}(\omega)$ electrical impedance of an oscillating piezoceramic disk involves solving the boundary problem of dynamic electroelasticity, which consists of three differential equations (5), (6) and (14) and boundary conditions (7), (8) and (15), (16). The link connecting the elastic and electrical parts of this problem is provided by equations (3) and (4) of the physical state of the piezoelectric.

The specific content of the physical state equations (3) and (4) is determined by constructing matrices of the piezoceramic material constants.

For a piezoceramic disk polarized along the z axis (in Fig. 1, we indicated the polarization direction with an arrow marked by P) the matrices of material constants are written as [20]:

– matrix of elastic moduli

$$\left|c^E_{\beta\lambda}\right| = \begin{vmatrix} c^E_{11} & c^E_{12} & c^E_{13} & 0 & 0 & 0 \\ & c^E_{22} & c^E_{23} & 0 & 0 & 0 \\ & & c^E_{33} & 0 & 0 & 0 \\ & & & c^E_{44} & 0 & 0 \\ & & & & c^E_{55} & 0 \\ & & & & & c^E_{66} \end{vmatrix}, \tag{17}$$

where β and λ are Voigt indices, each one combining a pair of tensor indices according to the following scheme $\beta \leftrightarrow i, j$ and $\lambda \leftrightarrow k, l$; one-to-one correspondence $1 \leftrightarrow (1,1)$; $2 \leftrightarrow (2,2)$; $3 \leftrightarrow (3,3)$; $4 \leftrightarrow (2,3;3,2)$; $5 \leftrightarrow (1,3;3,1)$ и $6 \leftrightarrow (1,2;2,1)$ remains between numeric values of Voigt indices ($\beta, \lambda = 1, 2, \ldots, 6$) and tensor indices ($i, j, k, l = 1, 2, 3$). The relationships between the numerical values of matrix elements are as follows (17): $c^E_{11} = c^E_{22} \neq c^E_{33}$; $c^E_{12} = c^E_{13} = c^E_{23}$; $c^E_{44} = c^E_{55}$; $c^E_{66} = \left(c^E_{11} - c^E_{12}\right)/2$;

– matrix of piezoelectric modules $e_{kij} \leftrightarrow e_k\beta$ (β is Voigt index)

$$|e_{k\beta}| = \begin{vmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{24} & 0 & 0 \\ e_{31} & e_{32} & e_{33} & 0 & 0 & 0 \end{vmatrix}, \tag{18}$$

where $e_{15} = e_{24}$; $e_{31} = e_{32} \neq e_{33}$; the study [19] found that piezo modules are $e_{15} = e_{24} = (e_{33} - e_{31})/2$ for piezoceramics;

– matrix of dielectric constants χ^{ε}_{kj}

$$\left|\chi^{\varepsilon}_{kj}\right| = \begin{vmatrix} \chi^{\varepsilon}_{11} & 0 & 0 \\ & \chi^{\varepsilon}_{22} & 0 \\ & & \chi^{\varepsilon}_{33} \end{vmatrix}, \tag{19}$$

where $\chi^{\varepsilon}_{11} = \chi^{\varepsilon}_{22} \neq \chi^{\varepsilon}_{33}$.

Here we consider the qualitative composition of the $\vec{D}(\rho, z)$ electrical induction vector. In the general (non-axisymmetric) case, the electric induction vector is composed with three components D_ρ, D_φ and D_z. In the problem under consideration, the physical state of the disk has axial symmetry, which is ensured by uniform electrode coating of the surfaces $z = 0$ and $z = \alpha$, therefore $D_\varphi \equiv 0$. Radial component of the electrical induction vector is $D_\rho = 2e_{15}\varepsilon_{\rho z} + \chi^\varepsilon_{11}E_\rho$. The radial component of the electric field strength vector is zero on $z = 0$ and $z = \alpha$ surfaces, as well as on the symmetry axis (Oz axis) of the disk and on the lateral surface $\rho = R$. On these surfaces, and on the Oz axis, the tangential stresses $\sigma_{\rho z} = 2c^E_{55}\varepsilon_{\rho z} - e_{15}E_\rho$ vanish, whence it follows that shear strains $\varepsilon_{\rho z} = \varepsilon_{z\rho}$ vanish on surfaces $\rho = R, z = 0, z = \alpha$ and on the Oz axis. Considering that shear strains $\varepsilon_{\rho z}$ and the radial component E_ρ of the electric field strength vector vanish simultaneously, we conclude that the radial component $D_\rho = 0$ on the surfaces $\rho = R, z = 0, z = \alpha$ and on the Oz axis. It may be demonstrated that in the volume of the disk there is $z > 0$ plane on which $\varepsilon_{\rho z}$ and E_ρ vanish, and the D_ρ component that vanishes consequently. If the disk is thin enough, then, taking into account the abundance of regions where $D_\rho = 0$, we can, as a first approximation, assume that $D_\rho = 0\,\forall\,(\rho,\, z) \in V$, where V is the disk volume.

Hence, the electric induction vector in thin disks is virtually completely determined by the axial component $D_z(\rho, z)$, which, as follows from condition (12), must satisfy the relation $\partial D_z/\partial z = 0$, which is equivalent to the statement that the axial component does not depend on z coordinate values, i. e., $D_z(\rho, z) \equiv D_z(\rho)$. It follows from definition (3) that

$$\begin{aligned} D_z(\rho) &= e_{31}\varepsilon_{\rho\rho} + e_{32}\varepsilon_{\varphi\varphi} + e_{33}\varepsilon_{zz} + \chi^\varepsilon_{33}E_z \equiv \\ &\equiv e_{31}\left(\frac{\partial u_\rho}{\partial \rho} + \frac{u_\rho}{\rho}\right) + e_{33}\frac{\partial u_z}{\partial z} - \chi^\varepsilon_{33}\frac{\partial \varphi}{\partial z} = \\ &= e_{31}\frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\rho u_\rho(\rho, z)\right] + e_{33}\frac{\partial u_z}{\partial z} - \chi^\varepsilon_{33}\frac{\partial \varphi}{\partial z}. \end{aligned} \tag{20}$$

In notation (20), we used the convention accepted in mechanics about designating numerically equal material constants e_{31} and e_{32} with identical symbols.

Resorting to the fact that the axial component of the electric induction vector does not depend on the values of the z coordinate, we integrate relation (20) over the z variable ranging from zero to α:

$$\begin{aligned} \alpha D_z(\rho) &= e_{31}\frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\rho \int_0^\alpha u_\rho(\rho, z)\, dz\right] + \\ &+ e_{33}\left[u_z(\rho,\alpha) - u_z(\rho, 0)\right] - \chi^\varepsilon_{33}\left[\varphi(\alpha) - \varphi(0)\right]. \end{aligned} \tag{21}$$

Further, we introduce the notation

$$u^{(z)}_\rho(\rho) = \frac{1}{\alpha}\int_0^\alpha u_\rho(\rho, z)\, dz, \tag{22}$$

and will denote value $u^{(z)}_\rho(\rho)$ as the radial component of the displacement vector of the disk's material particles, averaged over the disk's thickness. Since $\varphi(\alpha) - \varphi(0) \equiv U_0$, the expression (21) is therefore

$$\begin{aligned} D_z(\rho) &= e_{31}\frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\rho u^{(z)}_\rho(\rho)\right] + \\ &+ \frac{e_{33}}{\alpha}\left[u_z(\rho,\alpha) - u_z(\rho, 0)\right] - \chi^\varepsilon_{33}\frac{U_0}{\alpha}. \end{aligned} \tag{23}$$

Anticipating substitution of expression (23) into relation (2), we introduce the notation

$$u^{(\rho)}_z(z) = \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R \rho u_z(\rho, z)\, d\rho d\varphi = \frac{2}{R^2}\int_0^R \rho u_z(\rho, z)\, d\rho, \tag{24}$$

and will consider value $u^{(\rho)}_z(z)$ to be the axial component of the displacement vector of the disk material particles, averaged over the disk's thickness.

Substituting expression (23) into definition (2) of the electric charge Q on the $z = \alpha$ surface and considering definition (24), we obtain the following notation

$$\begin{aligned} Q &= 2\pi e_{31}Ru^{(z)}_\rho(R) + \\ &+ \frac{\pi R^2}{\alpha}e_{33}\left[u^{(\rho)}_z(\alpha) - u^{(\rho)}_z(0)\right] - \frac{\pi R^2}{\alpha}\chi^\varepsilon_{33}U_0. \end{aligned} \tag{25}$$

Here, we introduce the notation

$$C^\varepsilon_\partial = \frac{\pi R^2}{\alpha}\chi^\varepsilon_{33}, \tag{26}$$

and will consider value C^ε_∂ the dynamic electric capacitance of an oscillating piezoceramic disk. Definition (26) included, we can rewrite expression (25) as

$$Q = C^\varepsilon_\partial\,\Xi^{(\varepsilon)}(\omega), \tag{27}$$

where

$$\Xi^{(\varepsilon)}(\omega) = \frac{2e_{31}\alpha}{\chi^\varepsilon_{33}R}u^{(z)}_\rho(R) + \frac{e_{33}}{\chi^\varepsilon_{33}}\left[u^{(\rho)}_z(\alpha) - u^{(\rho)}_z(0)\right] - U_0. \tag{28}$$

Substituting relation (27) into definition (1), we obtain the expression for calculating the electrical impedance of the oscillating disk

$$Z_{el}(\omega) = \frac{U_0}{-i\omega C^\varepsilon_\partial\,\Xi^{(\varepsilon)}(\omega)}. \tag{29}$$

Expression (29) is valid in the high-frequency range, when both radial and axial displacements of the material particles exist simultaneously, i.e., when the length of the elastic wave becomes commensurate with the thickness of the disk.

Note that the disk's electrical impedance is determined by the averaged values of the displacement vector components of material particles. Thus, we can apply averaging operations (22) and (24) to equations (5) and (6), respectively, and thereby transform them into ordinary differential equations, which are always solvable with varying accuracy. It is important to notice that the system of partial differential equations (5) and (6) is fundamentally unsolvable in general form.

Within medium and low frequency range, the numerical values of the dynamic electric capacitance and the analytical design $\Xi^{(\varepsilon)}(\omega)$ alter due to the characteristic properties of the electric elastic state inherent in the oscillating disk.

3 Discussion of simulation results

Here, we will consider the electrical impedance of an oscillating piezoceramic disk in low frequency range.

Low frequencies will be the frequency range in which the length of the elastic wave (the scale unit of spatial inhomogeneity of the disk's stress-strain state) significantly exceeds the radial size of the disk R (by an order of magnitude or more). For example, for a disk with a diameter of 66 mm, the low frequency range will correspond to the range below 3 kilohertz. Hereby, mechanical stresses and elastic deformations do not change significantly, and at $\omega \to 0$, remain constant in the volume of the piezoceramic disk.

From the boundary conditions (7) and (8) it follows that normal stresses $\sigma_{\rho\rho}$ and consequently $\sigma_{\varphi\varphi}$ and σ_{zz} are equal to zero both on the surface and at any point in the volume of the piezoceramic disk. The same statement is true for shear stresses $\sigma_{\rho z}$, while condition $\sigma_{\rho z} = 0 \,\forall\, (\rho, z) \in V$ when $E_\rho = 0$ is equivalent to the condition $\varepsilon_{\rho z} = 0 \,\forall\, (\rho, z) \in V$. Under such assumptions, the generalized Hooke's law (4) implies a system of algebraic equations

$$\begin{aligned} c_{11}^E \varepsilon_{\rho\rho} + c_{12}^E \varepsilon_{\varphi\varphi} + c_{12}^E \varepsilon_{zz} &= e_{31} E_z, \\ c_{12}^E \varepsilon_{\rho\rho} + c_{11}^E \varepsilon_{\varphi\varphi} + c_{12}^E \varepsilon_{zz} &= e_{31} E_z, \\ c_{12}^E \varepsilon_{\rho\rho} + c_{12}^E \varepsilon_{\varphi\varphi} + c_{33}^E \varepsilon_{zz} &= e_{33} E_z. \end{aligned} \tag{30}$$

While writing the equation system (30), we will designate material constants of equal magnitude with identical symbols. Since normal and tangential stresses are equal to zero, equations (5) and (6) of steady-state harmonic oscillations in infinitely small piezoceramic volumes are satisfied with an error that is proportional to the unaccounted volumetric density of inertial forces. The solution to the equation system (30) with respect to the desired tension-compression strains $\varepsilon_{\rho\rho}, \varepsilon_{\varphi\varphi}$ and ε_{zz} is noted as

$$\begin{aligned} \varepsilon_{\rho\rho} = \varepsilon_{\varphi\varphi} &= \frac{e_{31} c_{33}^E - e_{33} c_{12}^E}{\left[c_{33}^E \left(c_{11}^E + c_{12}^E\right) - 2\left(c_{12}^E\right)^2\right]} E_z, \\ \varepsilon_{zz} &= \frac{e_{33} \left(c_{11}^E + c_{12}^E\right) - 2 e_{31} c_{12}^E}{\left[c_{33}^E \left(c_{11}^E + c_{12}^E\right) - 2\left(c_{12}^E\right)^2\right]} E_z. \end{aligned} \tag{31}$$

Substituting relations (31) into formula (20) brings us to

$$D_z = \chi_{33}^\sigma E_z = -\chi_{33}^\sigma \frac{U_0}{\alpha}, \tag{32}$$

where χ_{33}^σ is dielectric permittivity of piezoceramics in the mode of permanent mechanical stresses (equal to zero) in the volume and on the surface of the oscillating disk. This dielectric permittivity is calculated with the formula

$$\chi_{33}^\sigma = \chi_{33}^\varepsilon \left(1 + \Delta\chi_{33}^\sigma\right), \tag{33}$$

where

$$\Delta\chi_{33}^\sigma = \frac{2 e_{31}^2 c_{33}^E - 4 e_{31} e_{33} c_{12}^E + e_{33}^2 \left(c_{11}^E + c_{12}^E\right)}{\chi_{33}^\varepsilon \left[c_{33}^E \left(c_{11}^E + c_{12}^E\right) - 2\left(c_{12}^E\right)^2\right]}. \tag{34}$$

With values of material constants typical for PZT-type (lead zirconate titanate) piezoceramics, ($c_{11}^E = 110\,\text{GPa}$; $c_{12}^E = 60\,\text{GPa}$; $c_{33}^E = 100\,\text{GPa}$; $e_{33} = 18\,\text{C/m}^2$; $e_{31} = -8\,\text{C/m}^2$ and $\chi_{33}^\varepsilon = 1400\,\chi_0$; $\chi_0 = 8,85 \cdot 10^{-12}\,\text{F/m}$ is dielectric permittivity of vacuum, i.e., dielectric constant), calculation using formula (34) generates the following result: $\Delta\chi_{33}^\sigma = 0,844$, that is, dielectric constant in the low-frequency range (static dielectric constant), exceeds the high-frequency (dynamic) dielectric permittivity χ_{33}^ε almost two times.

When D_z is determined by formula (32), the function is $\Xi^{(\sigma)}(\omega) = -U_0$, and the dynamic, or, which can be more precisely defined, quasi-static electrical capacitance of the piezoceramic disk is $C_\partial^\sigma = \pi R^2 \chi_{33}^\sigma / \alpha$. Hereby, expression (29) takes form of a well-known formula for calculating the reactance of electrical capacitance $Z_{el}(\omega) = 1/(i\omega C_\partial^\sigma)$ and is confirmed by the high convergence between the theoretically obtained data and experimentally determined results, not exceeding 6%. For example, for a disk made of lead zirconate titanate PZT piezoelectric ceramics with a diameter of 66 mm and a thickness of 3 mm, the static capacitance was 16.68 nF.

Therefore, in the low frequency range, when the mechanical stresses in the piezoceramic disk are practically equal to zero, and the direct piezoelectric effect is practically absent, the electrical impedance of an oscillating piezoceramic disk takes form of the reactance of a capacitor with electrical capacitance C_∂^σ.

Conclusions

A mathematical model for ceramic disk-type piezoelectric transducers, which allows for the estimation of the electrical impedance and quasi-static electrical capacitance of these transducers in the operating frequency range, where the length of the elastic wave significantly (by an order of magnitude or more) exceeds the radial size of the disk, depending on their physico-mechanical characteristics has been developed.

Discovered analytical correlations enable the determination of parameters for the disk-type piezoceramic element, such as electrical impedance, piezoelectric modulus, dielectric constant, as well as electrical capacitance in the operating frequency range. It has been proven that the vector of electric induction in a thin piezoceramic disk is almost entirely determined by the axial component $D_z(\rho_z)$, which does not depend on the thickness of the disk. As a result, the problem of harmonic oscillations in the piezoelectric disk has been solved, contributing significantly to the simplification of calculations for such an element even at the design stage.

The static dielectric permeability of the piezoceramic disk in the low-frequency range has been determined, and, based on typical material constants (elastic modulus, piezoelectric modulus, and dielectric permeability coefficients) for the PZT-type piezoceramic, it is found that the value χ^{σ}_{33} exceeds the value of the high-frequency (dynamic) dielectric permeability χ^{ε}_{33} by a factor of 1.844.

It has been established that in the low-frequency range, when mechanical stresses in the piezoceramic disk approach zero and the direct piezoelectric effect is almost negligible, the electrical impedance of the oscillating piezoceramic disk is characterized as a reactive resistance of a capacitor with an electrical capacitance equivalent to the quasi-stationary electrical capacitance of the piezoceramic disk. This finding has been confirmed by a high convergence between the obtained theoretical data and experimentally determined results (with differences not exceeding 6%).

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Математичне моделювання електричного імпедансу п'єзокерамічного диска, що коливається в широкому діапазоні частот (Частина 1. Низькі частоти)

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В статті наводяться результати математичного моделювання та аналізу електричного імпедансу п'єзокерамічного диска, що здійснює коливання на низьких частотах, тобто коли довжина пружної хвилі суттєво (на порядок і більше) перевищує радіальний розмір диска. Так, запропонована математична модель дискових керамічних елементів п'єзоелектричних перетворювачів, які є важливим компонентом сучасних комунікаційних пристроїв, датчиків навколишнього середовища, прецизійного обладнання, медичних апаратів та іншого. Ключовою характеристикою математичної моделі, описаної у статті, є здатність визначати аналітичні залежності, які дозволяють оцінити такі основні електричні властивості п'єзокерамічного дискового елементу, як електричний імпеданс та квазістатичну електричну ємність, чим значно спростити розрахунок такого елементу ще на етапі його проєктування.

Досліджено статичну діелектричну проникність п'єзокерамічного диска, що коливається на низьких частотах. Вираховане значення такого параметра за значень фізичних констант, що характерні для п'єзокерамік сорту ЦТС (титанат цирконат свинцю) в 1,844 раза вище порівняно з показником високочастотної (динамічної) діелектричної проникності.

Виявлено, що у діапазоні низьких частот, коли механічні напруження в п'єзокерамічному диску, що коливається, наближаються до нульового рівня та прямий п'єзоелектричний ефект майже не відбувається, електричний імпеданс такого диска можна описати як реактивний опір конденсатора з електричною ємністю, еквівалентною квазістаціонарній ємності цього диска. Це підтверджується високим ступенем збігу теоретичних даних та результатів експериментів, де розбіжності не перевищують 6%.

Отримані в статті результати можуть бути корисними для наукових досліджень у галузях точного приладобудування та радіоапаратуробудування, а також для практичного застосування у розробці та виробництві високотехнологічного обладнання.

Ключові слова: п'єзоелектричний перетворювач; акустоелектроніка; математичне моделювання; імпеданс; дисковий елемент