*Visnyk NTUU KPI Seriia – Radiotekhnika Radioaparatobuduvannia*, 2024, Iss. 97, pp. 38–45

UDC 621.317.3:53.082.73

# Mathematical Modeling the Electrical Impedance of Piezoceramic Disk Oscillating in Wide Frequency Range (Part 2. Medium Frequencies)

*Bazilo C. V., Faure E. V., Usyk L. M., Tuz V. V., Chorniy A. M.*

Cherkasy State Technological University, Cherkasy, Ukraine

E-mail: *k.bazilo@chdtu.edu.ua*

This paper presents further developments in mathematical modeling of the electrical impedance of a piezoceramic disk in a wide frequency range, specifically focusing on the mid-frequency range, i.e., when the elastic wavelength becomes commensurate with the radius of the piezoceramic disk, which is important for numerous modern applications. A mathematical model was developed for disk piezoelectric transducers made of piezoceramics to estimate their electrical impedance and quasi-static electrical capacity in the medium frequency range basing on their geometrical, physical, and mechanical characteristics. The research has found that a piezoceramic disc attains electromechanical anti-resonance in the medium frequency range at frequency, at which its electrical impedance follows to infinity. This effect is due to the polarization charges being completely compensated by the electric charge, when the electric current vanishes and energy consumption from the generator is absent. The calculations proved that at frequencies close to the first thickness resonance (corresponding to the dimensionless wave number from 40 to 60), the radial displacements of material particles of the disk vanish. A very rapid decrease in the levels of radial shifts with a simultaneous increase in the electromechanical resonance number was noted. The evaluation of the mechanical quality factor of piezoceramic disk elements, obtained with the developed mathematical model, closely correlates with real values, which is confirmed by the high agreement between theoretical and experimental results.

*Keywords:* piezoelectric transducer; acoustoelectronics; mathematical modeling; impedance; disk element

DOI: 10.20535/RADAP.2024.97.38-45

## Introduction

Piezoceramic elements find multiple applications in modern technology due to their unique properties of high-precision conversion of mechanical energy into electrical energy and vice versa [1]. In particular, disk-shaped elements in the medium frequency range are extremely relevant in various fields, such as radio engineering, ultrasonic diagnostic systems, sensors and actuators [2, 3], as well as in communication and navigation technologies. Hence, piezoceramic disks are used in ultrasonic transducers for medical diagnostics, where operation accuracy and stability in medium frequencies are critical for obtaining high-quality images. In radioelectronic devices (especially in military equipment), such as filters and resonators, piezoceramic elements provide reliable and efficient operation in conditions of variable frequencies [4].

However, mathematical modeling of the electrical impedance of piezoceramic disks oscillating in the medium frequency range is currently a challenging task posed by the simultaneous action of interconnected mechanical and electrical processes within the material. Therefore, a difficult problem for the calculations is to present an accurate account of these processes and predict the impedance behavior under different operating conditions. On the other hand, traditional modeling methods cannot provide sufficient accuracy in every case, especially in conditions when the piezoceramic disk operates in modes close to resonance or anti-resonance frequencies [5], which is a significant challenge for engineers and researchers seeking to improve the efficiency and reliability of piezoceramic components.

In general, mathematical modeling of a piezoceramic disc's electrical impedance in the medium frequency range, as well as studying its behavior, is an indispensable step in developing radio engineering and functional instrumentation. Understanding the impedance behavior will enable creating more accurate models to optimize the design and control the operation of piezoceramic elements. This, in turn, will improve the quality and reliability of electronic devices in which these elements are used.

## 1 The relevance of the research based on the results of the publication analysis

A cohort of scientists and research organizations across the globe are engaged in the field of mathematical modeling of piezoceramic discs' electrical impedance, since this topic is relevant for the development of various technological fields.

The leading international research institutes include the Massachusetts Institute of Technology (MIT), Boston University, and The University of Texas at Austin.

The researchers at these institutions have made a significant contribution to the development of theoretical models and methods for analysing piezoceramic materials. For instance, Professor Ekinci K. L. [6] from Boston University is known for the developments in the field of piezoelectric materials and sensor technologies, and Professor Robert H. Bishop [7] from Austin (Texas) is engaged in improving the modeling methods of piezoceramic resonators and their application in electronics.

European research centers are also actively involved in this research direction. For example, scientists from European scientific institutions, such as the Max Planck Institute in Germany, the Imperial College of London, the Catholic University of Leuven in Belgium, and others conduct large-scale research on the properties of piezoceramic materials and their applications in various devices. Dr. Ilias Katsouras [8] from the Max-Planck Institute is one of the leading experts in the field of piezoelectric materials, and Professors J. Holterman and W. A. Groen [9] from Delft University of Technology (the Netherlands) specialise in mathematical modeling of the impedance piezoceramic elements.

In Asia, relevant research is being conducted by Fumio Narita [10], Professor in the Department of Frontier Sciences for Advanced Environment at Tohoku University in Japan. His current research focuses on deploying characterization techniques to penetrate into the fundamental structure-property relations of complex multifunctional composite materials.

In Ukraine, there are numerous academic communities that are vigorously researching mathematical modeling of the piezoceramic disks' electrical impedance. For example, the National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute" (KPI) is a leading Ukrainian research center in this field. Professor Y. M. Poplavko at the Department of Microelectronics is a well-known researcher in ferroelectric and dielectric physics, and the scientific group headed by him is engaged in the development of new modeling and analysis methods, including piezoceramic disks [11].

Another significant scientific center in Ukraine is the Institute for Problems of Material Sciences NAS of Ukraine (IPM NASU). The Institute conducts fundamental research on the properties of piezoceramic materials, as well as on the novel approaches to their mathematical modeling. Dr. G. S. Oleynyk is a leading academic at the Institute, who works on the theory and practice of using ceramic piezoceramic materials in various devices [12].

Thus, research in mathematical modeling of piezoceramic discs' electrical impedance has been intensely developed by numerous researchers and scientific organizations both abroad and in Ukraine. The results of these studies are definable for the development of radio engineering, instrument building, medical technology and many other fields that require accurate and reliable methods to analyze and control piezoelectric materials.

Therefore, mathematical modeling of a piezoceramic disc's electrical impedance in the medium frequency range is indispensably relevant and promising in view of the constant technological development and the growing demand for high-precision devices in various industrial sectors.

## 2 Formulation and solution of the problem of mathematical modeling a disc piezoceramic transducer that oscillates in the medium frequency range

Here, we present a disk with a radius $R$ many times greater than its thickness $\alpha$. The disk is located in a cylindrical coordinate system $\rho, \phi, z$, the origin of which is aligned with the center of its lower surface. The outer surfaces of the disk along the heights $z = 0$ and $z = \alpha$ are electrodes, i.e. surfaces covered with a thin layer of silver (up to 0.01 mm) with the thermal vacuum deposition technology described in a recent study [13]. The lower disk surface ($z = 0$) has a zero potential, since it is grounded, while electric potential $U_0 e^{i\omega t}$ ($U_0$ is the amplitude value of the electric potential) is supplied to the upper surface $z = \alpha$. The value for this potential is to be selected from the condition $U_0/\alpha << 0,1\,E_0$, where $E_0 \cong 2\,MV/m$ is the electric field strength in the polarizing material of the disk, which guarantees the absence of nonlinear effects; $i = \sqrt{-1}$ is the imaginary unit; $\omega$ is the angular sign inversion frequency of the potential; $t$ denotes time (Fig. 1).

We will accept medium frequencies as the frequency range within which the scale unit of spatial inhomogeneity of the stress-strain state (elastic wavelength) becomes commensurate with the radius of the piezoceramic disk. For thin disks, when the ratio is $\alpha/R << 1$, it follows from the above formulation that the stress-strain state remains practically unchangeable across the disk's thickness. From boundary conditions [14], it follows that $\sigma_{\rho z} = \sigma_{zz} = 0 \forall (\rho, z) \in V$. When $E_\rho = 0$, equation $\sigma_{\rho z} = 0$ leads to $\varepsilon_{\rho z} = \varepsilon_{z\rho} = 0 \forall (\rho, z) \in V$.

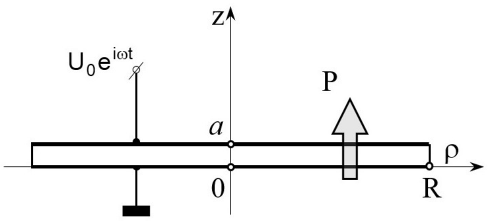


Fig. 1. Calculation diagram of a piezoceramic disk oscillating within the middle frequency range

Under the above assumptions, the generalized Hooke's law provides for

$$\sigma_{\rho\rho} = c_{11}^E \varepsilon_{\rho\rho} + c_{12}^E \varepsilon_{\phi\phi} + c_{12}^E \varepsilon_{zz} - e_{31} E_z, \quad (1)$$

$$\sigma_{\phi\phi} = c_{12}^E \varepsilon_{\rho\rho} + c_{11}^E \varepsilon_{\phi\phi} + c_{12}^E \varepsilon_{zz} - e_{31} E_z, \quad (2)$$

$$0 = c_{12}^E \varepsilon_{\rho\rho} + c_{12}^E \varepsilon_{\phi\phi} + c_{33}^E \varepsilon_{zz} - e_{33} E_z. \quad (3)$$

It follows from (3) that

$$\varepsilon_{zz} = -\frac{c_{12}^E}{c_{33}^E} (\varepsilon_{\rho\rho} + \varepsilon_{\phi\phi}) + \frac{e_{33}}{c_{33}^E} E_z. \quad (4)$$

Thus, substituting (4) between relations (1) and (2) yields

$$\sigma_{\rho\rho} = c_{11} \varepsilon_{\rho\rho} + c_{12} \varepsilon_{\phi\phi} - e_{31}^* E_z, \quad (5)$$

$$\sigma_{\phi\phi} = c_{12} \varepsilon_{\rho\rho} + c_{11} \varepsilon_{\phi\phi} - e_{31}^* E_z, \quad (6)$$

where $c_{11} = c_{11}^E - \left(c_{12}^E\right)^2 / c_{33}^E$ ; $c_{12} = c_{12}^E - \left(c_{12}^E\right)^2 / c_{33}^E$ ; $e_{31}^* = e_{31} - e_{33}/c_{33}^E$ are material constants for the planar oscillation mode, i. e., for the deformation mode when $\sigma_{zz} = 0$.

The component of the electrical induction vector $D_z = e_{31} (\varepsilon_{\rho\rho} + \varepsilon_{\phi\phi}) + e_{33} \varepsilon_{zz} + \chi_{33}^\varepsilon E_z$, after eliminating deformation $\varepsilon_{zz}$, is determined as

$$D_z = \frac{e_{31}^*}{\rho} \frac{\partial}{\partial \rho} \left[\rho\, u_\rho^{(z)}(\rho)\right] - \chi_{33}^* \frac{U_0}{\alpha}, \quad (7)$$

where $\chi_{33}^* = \chi_{33}^\varepsilon (1 + \Delta\chi_{33}^*)$ is dielectric constant for the planar oscillation mode. The addend $\Delta\chi_{33}^* = e_{33}^2 / \left(\chi_{33}^\varepsilon c_{33}^E\right)$ with material constant values typical for PZT-type ceramics [15] ($c_{11}^E = 110$ GPa; $c_{12}^E = 60$ GPa; $c_{33}^E = 100$ GPa; $e_{33} = 18$ C/m$^2$; $e_{31} = -8$ C/m$^2$ and $\chi_{33}^\varepsilon = 1400\, \chi_0$; $\chi_0 = 8,85 \cdot 10^{-12}$ F/m denote dielectric constant of vacuum or dielectric constant) does not exceed 0,262.

When determining component $D_z$ by expression (7), the formula calculating the electrical impedance of an oscillating disk reads

$$Z_{el}(\omega) = -\frac{U_0}{i\omega C_\partial^* \Xi^{(*)}(\omega)}, \quad (8)$$

where $C_d^* = \pi R^2 \chi_{33}^* / \alpha$ is the dynamic electrical capacitance of a piezoceramic disk for the planar oscillation mode, i.e., the electrical capacitance in the mid-frequency range. Function $\Xi^{(*)}(\omega)$ is calculated by the formula:

$$\Xi^{(*)}(\omega) = \frac{2 e_{31}^* \alpha}{\chi_{33}^* R} u_\rho^{(z)}(R) - U_0. \quad (9)$$

To determine the radial component of the disc's material particles' displacement vector averaged over the thickness of the disk at medium frequencies, we average the equation of steady radial oscillations [14]. Since

$$\frac{1}{\alpha} \int_0^\alpha \frac{\partial \sigma_{\rho z}}{\partial z} dz = \frac{1}{\alpha} [\sigma_{\rho z}(\rho, \alpha) - \sigma_{\rho z}(\rho, 0)] = 0,$$

the result of averaging can be represented by the following expression

$$\frac{\partial \sigma_{\rho\rho}^{(z)}(\rho)}{\partial \rho} + \frac{1}{\rho} \left[\sigma_{\rho\rho}^{(z)}(\rho) - \sigma_{\phi\phi}^{(z)}(\rho)\right] + \\ + \rho_0 \omega^2 u_\rho^{(z)}(\rho) = 0. \quad (10)$$

Normal stresses $\sigma_{\rho\rho}^{(z)}(\rho)$ and $\sigma_{\phi\phi}^{(z)}(\rho)$ are to be determined from the relations (5) and (6):

$$\sigma_{\rho\rho}^{(z)}(\rho) = \frac{1}{\alpha} \int_0^\alpha \sigma_{\rho\rho}(\rho)\, dz = \\ = c_{11} \frac{\partial u_\rho^{(z)}(\rho)}{\partial \rho} + c_{12} \frac{u_\rho^{(z)}(\rho)}{\rho} - e_{31}^* E_z^{(z)}(\rho), \quad (11)$$

$$\sigma_{\phi\phi}^{(z)}(\rho) = \frac{1}{\alpha} \int_0^\alpha \sigma_{\phi\phi}(\rho)\, dz = \\ = c_{12} \frac{\partial u_\rho^{(z)}(\rho)}{\partial \rho} + c_{11} \frac{u_\rho^{(z)}(\rho)}{\rho} - e_{31}^* E_z^{(z)}(\rho). \quad (12)$$

Since

$$D_z(\rho) = \frac{e_{31}^*}{\rho} \frac{\partial}{\partial \rho} \left[\rho\, u_\rho^{(z)}(\rho)\right] + \chi_{33}^* E_z^{(z)}(\rho), \quad (13)$$

it follows from comparing two physically equivalent definitions of the axial component of the electric induction vector (7) and (13) that

$$E_z^{(z)}(\rho) = -U_0/\alpha. \quad (14)$$

Substituting relations (11) and (12) into the ordinary differential equation (10) obtains the standard equation for Bessel functions

$$\rho^2 \frac{\partial^2 u_\rho^{(z)}(\rho)}{\partial \rho^2} + \rho \frac{\partial u_\rho^{(z)}(\rho)}{\partial \rho} + \left[(\lambda\rho)^2 - 1\right] u_\rho^{(z)}(\rho) = 0,$$

the solution for which consequently reads

$$u_{\rho}^{(z)}(\rho) = AJ_1(\lambda\rho), \tag{15}$$

where $A$ is the constant to be determined; $J_1(\lambda\rho)$ is first order Bessel function; $\lambda = \omega/\sqrt{c_{11}/\rho_0}$ is the wave number of radial vibrations in the piezoceramic disk.

The constant $A$ is determined from the boundary condition $\sigma_{\rho\rho}^{(z)}(R) = 0$ (while condition $\sigma_{\rho z}(R) = 0$ is inherently satisfied) as follows

$$A = -\frac{e_{31}^{*}U_0}{\alpha c_{11}}\frac{R}{[\lambda R\,J_0(\lambda R) - (1-k)\,J_1(\lambda R)]},$$

where $J_0(\lambda R)$ is zero order Bessel function; $k = c_{12}/c_{11}$ is a number less than one. Substituting the value of the constant $A$ into formula (15), and substituting the resulting expression into relation (9), we can rewrite expression (8) as:

$$Z_{el}(\omega) = \frac{1}{i\omega C_{\partial}^{*}}F^{(*)}(\omega), \tag{16}$$

where

$$F^{(*)}(\omega) = \frac{\lambda R\,J_0(\lambda R) - (1-k)\;J_1(\lambda R)}{\lambda R\,J_0(\lambda R) - (1-k-2K_{31}^2)\;J_1(\lambda R)}; \tag{17}$$

$K_{31}^2 = (e_{31}^{*})^2/(c_{11}\chi_{33}^{*})$ is square electromechanical coupling coefficient of the polarized through thickness piezoceramic disk in a radial oscillations' mode.

Function $F^{(*)}(\omega)$ determined by expression (17) has a defined number of characteristic points on the $\omega$ frequency axis. At $\omega_{rm}$ frequency, which corresponds to the $m$-th root of the equation,

$$x\,J_0(x) - (1-k)\;J_1(x) = 0, \tag{18}$$

the function is $F^{(*)}(\omega_{rm}) = 0$. At $\omega_{rm}$ frequency, the sign inversion frequency of the Coulomb forces deforming the piezoceramic disk coincides with the $m$-th natural frequency of the radial axisymmetric vibrations of the disk and there occurs resonant energy consumption from the source of elastic vibrations. Since an ideal source of a harmonically time-varying electrical potential difference has an infinite supply of energy at frequencies of radial electromechanical resonances, the amplitude values of the $u_{\rho}^{(z)}(\rho)$ radial component of the disk's material particles displacement vector increase indefinitely. The amplitude values of the strain tensor components increase accordingly, and, as a consequence, the amplitude values of the electric charge on the surface $z = \alpha$ increase indefinitely. The latter induces an unlimited growth of amplitude values of alternating current in the conductors that are connected to the electrode coating of the disk. The infinite currents at the output of an ideal generator of electric voltage arise as a result of a short circuit, i.e., when the load resistance is $Z_{el}(\omega_{rm}) = 0$.

Table 1 presents numerical values of the first two roots ($x_1$ and $x_2$) of equation (18) and their $\xi_{21} = x_2/x_1$ ratio depending on the $k$-parameter.

Note that the ratio $\xi_{21}$ is equal to the ratio $\omega_{21} = \omega_{r2}/\omega_{r1}$, i.e., the circular frequencies of the second and first radial electromechanical resonances, which can be easily and accurately determined experimentally. Note also that the numerical values of the equation roots (18) change insignificantly compared to the change in the $k$-parameter. This must be considered when performing measurements, which must be performed with all possible care.

Table 1 The first two roots of the equation $x\,J_0(x) - (1-k)\;J_1(x) = 0$.

| $k$ | $x_1$ | $x_2$ | $\xi_{21}$ |
|---|---|---|---|
| 0,00 | 1,841184 | 5,331443 | 2,895660 |
| 0,05 | 1,878980 | 5,341153 | 2,842582 |
| 0,10 | 1,915393 | 5,350843 | 2,793601 |
| 0,15 | 1,950511 | 5,360511 | 2,748259 |
| 0,20 | 1,984414 | 5,370155 | 2,706167 |
| 0,25 | 2,017172 | 5,379773 | 2,666988 |
| 0,30 | 2,048850 | 5,389364 | 2,630434 |
| 0,35 | 2,079508 | 5,398928 | 2,596253 |
| 0,40 | 2,109198 | 5,408462 | 2,564226 |
| 0,45 | 2,137971 | 5,417963 | 2,534162 |
| 0,50 | 2,165871 | 5,427433 | 2,505889 |
| 0,55 | 2,192942 | 5,436869 | 2,479259 |
| 0,60 | 2,219221 | 5,446270 | 2,454137 |
| 0,65 | 2,244744 | 5,455635 | 2,430404 |
| 0,70 | 2,269547 | 5,464962 | 2,407953 |
| 0,75 | 2,293658 | 5,474251 | 2,386690 |
| 0,80 | 2,317109 | 5,483500 | 2,366527 |
| 0,85 | 2,339926 | 5,492708 | 2,347385 |
| 0,90 | 2,362135 | 5,501874 | 2,329195 |
| 0,95 | 2,383761 | 5,510998 | 2,311892 |
| 1,00 | 2,404826 | 5,520078 | 2,295417 |

The $\omega_{rm}$ frequencies of radial resonances are followed by the frequencies at which the denominator of expression (17) vanishes. At these frequencies, the $F^{(*)}(\omega)$ function increases infinitely. Accordingly, the electrical impedance $Z_{el}(\omega)$ of the piezoceramic disk increases indefinitely since the polarization charges completely compensate for the electrical charge that the electrical signal generator generates on the electrode coating of the disc. The net charge $Q$ vanishes, and the electric current in the conductors disappears. This corresponds to an open electrical circuit or an electrical circuit in which an infinitely large resistance is included. In this case, naturally, the piezoceramic disk does not consume energy from the oscillation source, i.e., from the generator. To emphasize the specificity and difference of this state from the state at the frequencies of electromechanical resonance, the $\omega_{am}$ frequencies where $Z_{el}(\omega_{am}) \rightarrow \infty$ are called the frequencies of electromechanical antiresonance.

In other words, at the frequencies of electromechanical antiresonance $\omega_{am}$, the electrical impedance of the piezoceramic disc $Z_{el}(\omega)$ increases indefinitely, and the electric current in the conductors vanishes, which corresponds to the state of an open circuit.

Thus, in the medium frequency range, similarly to the low frequency range, there is a change in the numerical values of the dynamic electrical capacity and analytical design due to the peculiarities of the electro-elastic state of the oscillating disk $\Xi^{(\varepsilon)}(\omega)$.

## 3 Discussing of the modeling results

In a real experiment, there are no zeros or infinities, since in real elastic materials there are always viscous friction losses. These losses can be calculated through parameter $Q$, which has the meaning of the mechanical quality factor of the material. The $Q$-factor is a dimensionless number, the value of which is inversely proportional to the energy losses in the oscillatory system per period. In ideal elastic bodies, where viscous friction entails no energy loss, $Q \to \infty$. In real objects, the $Q$ quality factor has a finite value. Thus, the elasticity moduli $c^E_{\beta\lambda}(Q)$ read as follows [16]

$$c^E_{\beta\lambda}(Q) = c^E_{\beta\lambda}(1 + i/Q), \qquad (19)$$

where $c^E_{\beta\lambda}$ is the static modulus of elasticity; $i = \sqrt{-1}$ is unit imaginary number.

Figure 2 demonstrates radial displacement modules $u^{(z)}_\rho(\rho)$ in a piezoceramic disk with radius $R = 33 \times 10^{-3}$ м and thickness $\alpha = 3{\cdot}10^{-3}$ м. Disc material (piezoceramic) parameters are as follows: $c^E_{11} = 110$ GPa; $c^E_{12} = 60$ GPa; $c^E_{33} = 100$ GPa; $e_{33} = 18\,\mathrm{C/m^2}$; $e_{31} = -8\,\mathrm{C/m^2}$ и $\chi^\varepsilon_{33} = 1400\,\chi_0$; $Q = 100$; $k = c_{12}/c_{11} = 0,324$. The calculations were performed at the frequencies of the first three electromechanical resonances. The resonant frequency number is indicated in the figure field next to the corresponding curve. For the above $k$-parameter value, the following values of the equation roots (18) correspond to the resonant frequencies: $x_1 = 2,063690$; $x_2 = 5,393958$ and $x_3 = 8,574693$. The electric potential is $U_0 = 1$ V. We have plotted the values of the dimensionless radial coordinate $\rho/R$ along the abscissa axis in Figure 2.

Noteworthy is the extremely rapid decrease in the levels of radial displacements where the order of electromechanical resonance increases. This fact is further illustrated by Figure 3, which shows the change in the radial displacement modulus $u^{(z)}_\rho(R)$ of the piezoceramic disk lateral surface $\rho = R$ in a wide frequency range. The calculations were performed based on the above set of geometric, physical and mechanical parameters of the oscillating disk. The numbers in the figure field indicate the numbers of electromechanical resonances. Along the abscissa axis in Figure 3, the dimensionless wave number $x = \lambda R$ is plotted. From the calculation results presented in Figure 3, it follows that at frequencies in the vicinity of the first thickness resonance (approximately corresponding to the values $\lambda R \cong 40 \div 60$), the radial displacements of the disc's material particles, being calculated under the assumption that $\sigma_{zz} = 0$, cease to exist.

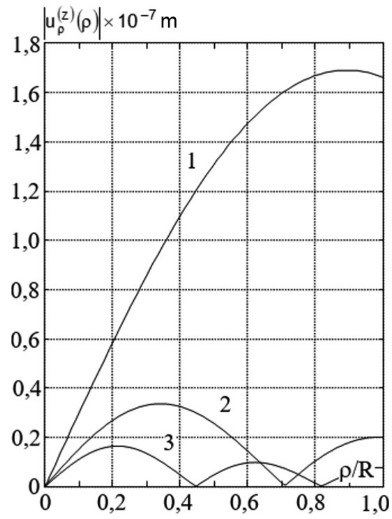


Fig. 2. Radial displacements of the piezoceramic disk material particles at the frequencies of the first three electromechanical resonancese

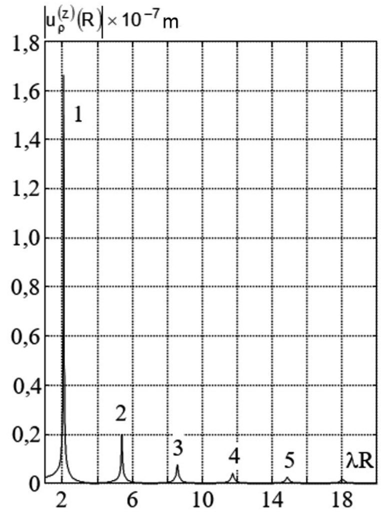


Fig. 3. Radial displacements of the piezoceramic disk's lateral surface in a wide frequency range

Figure 4 demonstrates the change in the electrical impedance modulus of a piezoceramic disk in the mid-frequency range (Fig. 4a). The inset in Figure 4a shows the change in $Z_{el}(\omega)$ modulus values in the immediate vicinity of the first electromechanical resonance frequency $\omega_{r1}$. Figure 4b reflects the change in the electrical impedance modulus in the vicinity of the of the first electromechanical antiresonance frequency $\omega_{a1}$. The disk's geometric and physical mechanical parameters serving as a base for calculations according to formula (16), are indicated in the comments to Figure 3.

Formula (19) implies that expressions (16) and (17) are functions of a small parameter $\epsilon = 1/Q$. Expanding the function $Z_{el}(\omega)$ into a power series in a small parameter $\epsilon$, and limiting the expansion to the first power of this parameter, we obtain the following estimate of the piezoceramic mechanical quality factor [17]:

$$Q = \frac{R\left[\left(x_m^2 + k - 1\right) J_1(x_m) - k J_0(x_m)\right]}{4 x_m v C_\partial^*(0)\, K_{31}^2 J_1(x_m)\, Z_{el}(x_m)}, \qquad (20)$$

where $x_m$ is the $m$-th root of the equation (18); $v = \sqrt{\mathrm{Re}\, c_{11}/\rho_0}$ is the propagation speed of radial vibrations in piezoceramics, determined ignoring the losses caused by viscous friction; $C_\partial^*(0)$ is the dynamic electrical capacitance of an oscillating disk determined through the dielectric constant $\mathrm{Re}\,\chi_{33}^*$ ignoring the losses caused by viscous friction; $Z_{el}(x_m)$ is the electrical impedance of the disk at the $m$-th frequency of electromechanical resonance. Note that $Z_{el}(x_m)$ is a real value. From the inset in Figure 4a $Z_{el}(x_1) = 7,8$ Ohms is determined. Substituting this quantity into formula (20) produces value $Q = 100,096$. The value $Q = 100$ was included in the calculation, i.e., the resulting estimate is in good agreement with the true value of the mechanical quality factor, and the discrepancy between theoretically obtained data and experimentally determined results was less than $10^{-3}\%$.

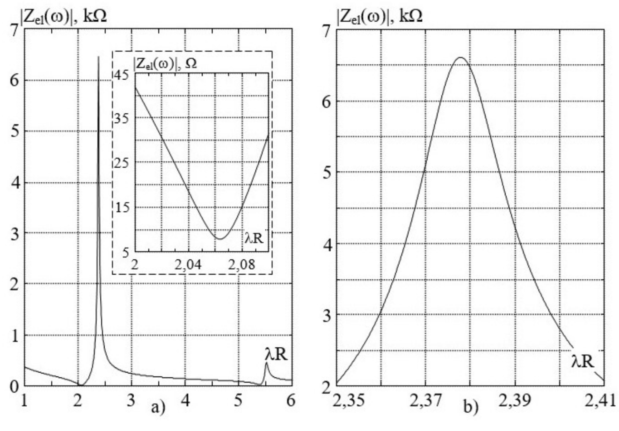


Fig. 4. Module of electrical impedance of the disk in the mid-frequency range

Several studies attempt to determine the mechanical quality factor with the ratio $\omega_{a1}/\Delta\omega_{0,707}$, where $\omega_{a1}$ is the angular frequency of the first electromagnetic resonance; $\Delta\omega_{0,707}$ is the frequency range at the level of –3 dB from the modulus value $Z_{el}(\omega_{a1})$, i. e., maximum value of electrical impedance at the frequency of the first antiresonance. Performing the calculations according to the graph shown in Figure 4b produces $Q = 153,3$ value that differs significantly from the $Q = 100$ value included in the calculation. This discrepancy is linked to the electromechanical anti-resonance which appears as a consequence of the interaction (mutual compensation) of the electric charge, which is induced by the electric potential difference generator on the electrode coating of the disk, and the polarization charge, which emerges as a result of the piezoelectric deformation. In other words, unlike electromechanical resonance, viscous friction losses are by no means the only factor determining the magnitude of the electrical impedance.

Figure 5 demonstrates the frequency-dependent change occurring in the modulus of the dimensionless function $F^{(*)}(\omega)$, which is defined by expression (17). This calculation was performed for the same set of geometric, physical, and mechanical parameters included into the calculations presented by Figure 3 and Figure 4. The inset in the Figure 5 field shows the change in the $F^{(*)}(\omega)$ function modulus within the range of dimensionless wave numbers $3 \le \lambda R \le 5$ or, equally, dimensionless frequencies $3 \le \omega\tau_0 \le 5$ ($\tau_0 = R/v$ ), where $\left|F^{(*)}(\omega)\right| \approx 1$. With $\lambda^* R = \omega^*\tau_0 = 3,83$ values, the $F^{(*)}(\omega)$ function modulus is equal to unity and the electrical impedance is defined as $|Z_{el}(\omega^*)| = 1/\omega^*\left|C_\partial^*\right|$ . Known $|Z_{el}(\omega^*)|$ value enables us to determine the modulus of dynamic electrical capacitance in the mode of planar oscillations of the piezoceramic disk, and, as a result, to obtain an estimate of the dielectric constant modulus $\chi_{33}^*$. Since while performing measurements of $\omega^*$ frequency at which $\left|F^{(*)}(\omega^*)\right| = 1$ is a priori unknown, its value, to a first approximation, can be estimated as follows: $\omega^* \cong (\omega_r + \omega_a)/2$ , where $\omega_r = (\omega_{r1} + \omega_{r2})/2$ and $\omega_a = (\omega_{a1} + \omega_{a2})/2$ .

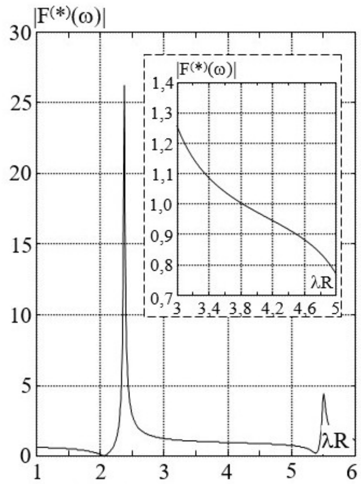


Fig. 5. Frequency-dependent alteration in the modulus of function $F^{(*)}(\omega)$

## Conclusions

The study offers a mathematical model developed for disc piezoelectric transducers made of piezoceramics. The model makes it possible to estimate the electrical impedance of the disk transducers studied here and their quasi-static electrical capacity in the mid-frequency range depending on their geometrical and physical-mechanical parameters.

The study has shown that a piezoceramic disc operating in the medium frequency range, i.e., when the elastic wavelength becomes commensurate with the radius of the piezoceramic disk, attains electromechanical anti-resonance at a $\omega_{am}$ frequency, when its electrical impedance is $Z_{el}(\omega_{am}) \rightarrow \infty$. The specific property of this state is that under these conditions, the polarization charges completely compensate the electric charge, hence the electric current vanishes and energy consumption from the generator drops to zero.

The calculations have proved that at frequencies in the vicinity of the first thickness resonance (approximately corresponding to the wave number values in the range of 40-60), the radial displacements of the material particles of the disc cease to exist under the $\sigma_{zz} = 0$ condition. At the same time, the research has revealed an extremely rapid decrease in the radial shift levels as the number of electromechanical resonances increases.

The estimate of the mechanical quality factor of piezoceramic disk elements, obtained after calculating the mathematical model, highly correlates with the real values of the mechanical quality factor. The established fact was confirmed by a high degree of agreement between theoretically obtained data and experimentally determined results (the discrepancy was less than $10^{-3}$%).

The data presented in this paper have been obtained as implementation of the experimental scientific and technical project titled “Development of an automated ultrasonic system for extracting plant raw materials to produce multi-nutrient functional drinks for rehabilitation and preventing post-traumatic stress disorders” (national registration number: 0124U000713, 2024-2025), which is being carried out at the Cherkasy State Technological University.

# References

[1] Erhart J., Půlpán P., Pustka M. (2017). *Piezoelectric Ceramic Resonators*. Springer, Cham, Switzerland. DOI: 10.1007/978-3-319-42481-1.

[2] Bazilo C. (2020). Modelling of bimorph piezoelectric elements for biomedical devices. In: Hu Z., Petoukhov S., He M. (eds) *Advances in Artificial Systems for Medicine and Education III. Advances in Intelligent Systems and Computing*, Vol. 1126, Springer, Cham, pp. 151–160. DOI: 10.1007/978-3-030-39162-1_14.

[3] Aladwan I. M., Bazilo C., Faure E. (2022). Modelling and Development of Multisectional Disk Piezoelectric Transducers for Critical Application Systems. *Jordan Journal of Mechanical and Industrial Engineering*, Vol. 16, No. 2, pp. 275–282.

[4] Behera A. (2022). Piezoelectric Materials. In: *Advanced Materials*. Springer, Cham, Switzerland, pp. 43–76. DOI: 10.1007/978-3-030-80359-9_2.

[5] Yazdian A., Karafi M. R. (2023). An analytical model to study the frequency response of ultrasonic welding transducers. *SN Appl. Sci.*, Vol. 5, no. 157. DOI:10.1007/s42452-023-05368-x.

[6] Kouh T., Hanay M. S., Ekinci K. L. (2017). Nanomechanical Motion Transducers for Miniaturized Mechanical Systems. *Micromachines*. Vol. 8, no.4. DOI: 10.3390/mi8040108.

[7] Bishop R. H. (2002). *The Mechatronics Handbook*. CRC Press, Washington, D. C.

[8] Katsouras I., Asadi K., Li M., van Driel T.B., Kjær K.S., et al. (2016). The negative piezoelectric effect of the ferroelectric polymer poly (vinylidene fluoride). *Nat Mater.*, Vol. 15(1), pp. 78-84. DOI: 10.1038/nmat4423.

[9] Holterman J., Groen W.A. (2012). *An introduction to piezoelectric materials and components*. Stichting Apllied Piezo, Apeldoorn.

[10] Narita F., Wang Zh. (2024). *Piezoelectric Materials, Composites, and Devices: Fundamentals, Mechanics, and Applications*. Academic Press, UK.

[11] Poplavko Y. M., Yakimenko Y. I. (2013). *Piezoelectrics*. National Technical University of Ukraine “Igor Sikorsky Kyiv Polytechnic Institute”, Kyiv, Ukraine.

[12] Oleynyk G. S. (2018). *Structure formation of ceramic materials*. Naukova dumka, Kyiv, Ukraine.

[13] Antonyuk V. S., Bondarenko M. O., Bondarenko Yu. Yu. (2012). Studies of thin wear-resistant carbon coatings and structures formed by thermal evaporation in a vacuum on piezoceramic materials. *Journal of Superhard Materials*, Vol. 34, no. 4. DOI: 10.3103/ S1063457612040065.

[14] Petrishchev O. N. (2012). *Harmonic vibrations of piezoceramic elements. Part 1. Harmonic vibrations of piezoceramic elements in vacuum and the method of resonance-antiresonance*. Avers, Kiev, Ukraine.

[15] Kathavate V. S., Prasad K. E., Kiran M. S. R. N., Zhu Y. (2022). Mechanical characterization of piezoelectric materials: A perspective on deformation behavior across different microstructural length scales. *J. Appl. Phys.*, Vol. 132, no. 12. DOI: 10.1063/5.0099161.

[16] Bardzokas D. I., Filshtinsky M. L., Filshtinsky L. A. (2007). Mathematical Methods in Electro-Magneto-Elasticity. In: *Lecture Notes in Applied and Computational Mechanics*. Vol. 32, 2007th Ed. Springer, USA.

[17] Bazilo C. V. (2018). Principles and methods of the calculation of transfer characteristics of disk piezoelectric transformers. *Radio Electronics, Computer Science, Control*, no. 4, pp. 7-22. DOI: 10.15588/1607-3274-2018-4-1.

## Математичне моделювання електричного імпедансу п’єзокерамічного диска, що коливається в широкому діапазоні частот (Частина 2. Середні частоти)

*Базіло К. В., Фауре Е. В., Усик Л. М., Туз В. В., Чорній А. М.*

У даній статті представлено подальше дослідження з математичного моделювання електричного імпедансу п’єзокерамічного диска в широкому спектрі частот, з особливою увагою до середньочастотного діапазону, тобто коли довжина пружної хвилі стає співмірною з радіусом п’єзокерамічного диска, який є важливим для численних сучасних застосувань. Розроблена математична модель для п’єзоелектричних перетворювачів дискової форми з п’єзокераміки, що дозволяє оцінити їх електричний імпеданс та квазістатичну електричну ємність в середньочастотній області, залежно від геометричних та фізико-механічних характеристик таких перетворювачів. Встановлено, що п’єзокерамічний диск у середньочастотній області досягає стану електромеханічного антирезонансу на частоті, на якій його електричний імпеданс прямує до нескінченості. Це відбувається через повну компенсацію поляризаційних зарядів електричним зарядом, що призводить до зникнення електричного струму та відсутності споживання енергії від генератора. Розрахунки показали, що на частотах поблизу першого товщинного резонансу (відповідає безрозмірному хвильовому числу від 40 до 60), радіальні зміщення матеріальних частинок диска зникають. Відзначено дуже швидке зменшення рівнів радіальних зсувів зі збільшенням номера електромеханічного резонансу. При цьому, оцінка механічної добротності п’єзокерамічних дискових елементів, отримана за допомогою математичної моделі, тісно корелює з реальними значеннями, що підтверджено високою збіжністю між теоретичними та експериментальними результатами.

*Ключові слова:* п’єзоелектричний перетворювач; акустоелектроніка; математичне моделювання; імпеданс; дисковий елемент