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Coupled Oscillations of Rectangular Lattices of Dielectric Resonators (Analytical Solutions)

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The frequency spectra and amplitude distributions of the natural oscillation fields of the same shape and dimensions coupled dielectric resonator (DR) systems, located in one-, two-, and three-dimensional rectangular lattices, are considered. Neglecting the coupling between non-adjacent resonators, general analytical solutions are found for the linear homogeneous system of equations proposed earlier for describing natural oscillations of coupled DR systems. An algorithm based on perturbation theory for solving systems of equations of coupled oscillations of identical DRs is proposed, which allows reducing the solution to the calculation of determinants of tridiagonal and pentadiagonal matrices. It's shown that parameters of coupled oscillations of rectangular structures are determined through simple distributions of amplitudes and frequencies characteristic of Bloch waves of quantum particles in a periodic potential. Using the derived general analytical formulas, the calculated frequencies are compared with the natural oscillation frequencies obtained by numerical methods. For the first time, a general analytical solution is found for the distribution of amplitudes and frequencies of a rectangular lattice with DR doubly degenerate types of natural oscillations. It's shown that in the case of zero coupling between degenerate oscillations of different types, the obtained analytical formulas transform into expressions describing the oscillations of DRs in simple rectangular lattices. General conditions are formulated, under which the solution of the equations for coupled oscillations of DR systems can be found in analytical form. It's shown that under the specified conditions, the distribution of the amplitudes of coupled oscillations of identical DR lattices with degenerate and non-degenerate types of eigenoscillations are interconnected to each other. In this case, a new method for calculating the amplitudes and frequencies of coupled oscillations of DR lattices with degenerate oscillations is proposed. The obtained formulas allow us to estimate in general terms the characteristics of the spectrum of eigenoscillations with an increasing in the number of resonators. Several examples demonstrate a very good coincidence of the found analytical and numerical results. The obtained theoretical conclusions significantly simplify the calculation and optimization of scattering parameters of various communication devices in the microwave, infrared and optical wavelength ranges, which are built based on the use of rectangular DR structures.

Keywords: dielectric resonator; coupled oscillations; rectangular lattice; spectral theory

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Introduction

Rectangular lattices of dielectric resonators (DRs) [1–22] are applied in various devices in the terahertz, infrared and optical wavelength ranges, as well as in filters [10, 16–22]; modulators [3–5]; lasers [1, 6] and so on. The most widely used are one-dimensional [15–17, 19, 21, 22] and two-dimensional [3–7, 12–14, 18], and more recently three-dimensional lattices [1, 2, 5, 8–12, 20]. Such lattices contain a large number of DRs, which complicates optimization of their scattering parameters. Usually, the calculation of parameters of the lattice is performed by using numerical methods, which requires significant computer resources. Meanwhile, in some cases, it's possible to obtain more simple analytical solutions to problems of natural oscillations for systems particularly containing a big number DRs. Obtaining analytical expressions for such complex lattices of coupled DRs allows us to significantly simplify their analysis and optimization.

1 Statement of the problem

The purpose of this article is to obtaining analytical solutions and analyzes problems of coupled oscillations of one-, two-, and three-dimensional rectangulare lattices of DRs with no degenerate and degenerate oscillations.

2 Natural oscillations of DR lattices

In the [25] it's looked for a solution to the problem of coupled oscillations of a system of N DRs obtained in the form of an expansion their field $(\mathbf{e}, \mathbf{h})$ in terms of the natural oscillations of the same, but isolated resonators $(\mathbf{e}_s, \mathbf{h}_s)$:

$$\begin{pmatrix} \mathbf{e} \\ \mathbf{h} \end{pmatrix} = \sum_{\mathrm{s}=1}^{\mathrm{N}} \mathrm{b_s} \begin{pmatrix} \mathbf{e}_\mathrm{s} \\ \mathbf{h}_\mathrm{s} \end{pmatrix}. \qquad (1)$$

In general was obtained equation system for amplitudes $\|\mathrm{b}_s\|$ (1):

$$\sum_{\mathrm{s}=1}^{\mathrm{N}} \kappa_{\mathrm{st}}\, \mathrm{b_s} - \lambda \mathrm{b}_t = 0 \quad (s, t = 1,\ 2, \ldots,\ N)\,, \qquad (2)$$

where

$$\lambda\ = 2(\tilde{\omega} - \omega_0)/\omega_0 =\ 2\,(\delta\omega/\omega_0 + \mathrm{i}\ \omega''/\omega_0)\,; \qquad (3)$$

$\tilde{\omega}$ – complex frequency of coupled oscillations; $\delta\omega = \mathrm{Re}(\tilde{\omega}-\omega_0)$; $\omega'' = \mathrm{Im}(\tilde{\omega})$; ω_0 – real part of the frequency of isolated DRs.

The distribution of the amplitudes of coupled oscillations of a system of identical resonators $\|\mathrm{b}_s\|$ was formulated as an eigenvalues problem for a finite-dimensional coupling operator K$= \|\kappa_{\mathrm{st}}\|$:

$$K = \begin{pmatrix} i\tilde{k}_1 & \kappa_{21} & \kappa_{31} & \cdots & \kappa_{N1} \\ \kappa_{12} & i\tilde{k}_2 & \kappa_{32} & \cdots & \kappa_{N2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \kappa_{1,N-1} & \kappa_{2,N-1} & \kappa_{3,N-1} & \cdots & \kappa_{N,N-1} \\ \kappa_{1,N} & \kappa_{2,N} & \kappa_{3,N} & \cdots & i\tilde{k}_N \end{pmatrix}, \qquad (4)$$

where κ_{st} are coupling coefficients of a s–th and t–th DR. Diagonal elements of the coupling operator matrix K determined only by the magnitude of the radiation of s-th partial resonators, represented by coupling coefficient $\tilde{k}_s$.

Equating to zero, the determinant of the system (2),

$$\det \left\| \kappa_{\mathrm{st}}(1 - \delta_{st}) + (i\tilde{k}_s - \lambda)\delta_{st} \right\| = 0, \qquad (5)$$

was obtained the characteristic equation, the solution of which determines the frequency splitting that arises due to the electromagnetic influence of the resonators. In this case, each non-degenerate value of the frequency $\tilde{\omega}^s = \omega^s + i\omega^{s\prime\prime}$ $(s = 1, 2, \ldots, N)$ of the system s-th natural oscillations corresponds to its own column vector:

$$\mathbf{b}^s = \|\mathrm{b}_\mathrm{t}^\mathrm{s}\| = \begin{pmatrix} \mathrm{b}_1^\mathrm{s} \\ \mathrm{b}_2^\mathrm{s} \\ \vdots \\ \mathrm{b}_N^\mathrm{s} \end{pmatrix}, \quad (s = 1, 2, \ldots, N) \qquad (6)$$

of the coupling operator K $(t = 1, 2, \ldots, N)$ (4), determining the distribution of amplitudes of partial resonators. Thus, in the absence of degeneracy of the DR natural oscillations, a system consisting of N resonators is characterized by a $N \times N$ matrix of amplitudes of coupled oscillations:

$$B = \begin{pmatrix} b_1^1 & b_1^2 & \cdots & b_1^N \\ b_2^1 & b_2^2 & \cdots & b_2^N \\ \vdots & \vdots & \vdots & \vdots \\ b_N^1 & b_N^2 & \cdots & b_N^N \end{pmatrix}. \qquad (7)$$

In the general case, the solution of the system of equations (2) is carried out numerically, however, in some special cases they can be found in analytical form, which significantly increases the speed of calculations, especially for large systems $(N \gg 1)$. We have considered particular solutions of (2) for different rectangular lattices of identical DRs ($\tilde{k}_s = \tilde{k}_0$) $(s = 1, 2, \ldots, N)$.

3 Natural oscillations of one-dimensional lattices DRs

To find the eigenvalues and eigenfunctions of the DR system, a relatively simple case was considered, when each resonator is coupled only to its neighbors. Let us call it the first approximation (Fig. 1, a). The second approximation determined the solution obtained under the condition of taking into account the coupling of resonators with neighbors and neighboring neighbors (Fig. 1, b). Continuing this process, if desired, we can calculate more accurately the parameters of the lattice's natural oscillations.

At first, we considered the simple case of a one-dimensional lattice of N identical DRs, in which all resonators are coupled only to their neighbors. Under this condition, the system of equations (1) takes the form:

$$\kappa_{\text{t-1,t}}\mathrm{b}_{t-1} + (i\tilde{k}_t - \lambda)\mathrm{b}_t + \kappa_{\text{t+1,t}}\mathrm{b}_{t+1} = 0,$$
$$(t = 1, 2, \ldots, N).$$

For identical DRs, the coupling coefficients of the resonators with open space are equal to each other: $\tilde{k}_t = \tilde{k}_0$. And if the resonators are located at equal distances from each other: $\kappa_{t-1,t} = \kappa_{t+1,t} = \kappa_{12}$ then the system is simplified

$$\kappa_{12}\mathrm{b}_{t-1} + (i\tilde{k}_0 - \lambda)\mathrm{b}_t + \kappa_{12}\mathrm{b}_{t+1} = 0. \qquad (8)$$

The solution of system (8) is well known [23]; we represented it as a set of normalized eigenvectors with an amplitude distribution:

$$b_t = b_0 \sin(\theta t), \quad (t = 1, 2, \ldots, N). \qquad (9)$$

Substituting (9) into equations (8), we obtain:

$$2\kappa_{12}\cos\theta + (i\tilde{k}_0 - \lambda) = 0. \qquad (10)$$

We supplemented equations (9), (10) with the condition of symmetry of the amplitude distribution for all resonators of the lattice $|b_v| = |b_{N-v+1}|$ $(v = 1, 2, \ldots, N)$; from which we found:

$$|\sin(\theta v)| = |\sin[\theta(N - v + 1)]|\,. \qquad (11)$$

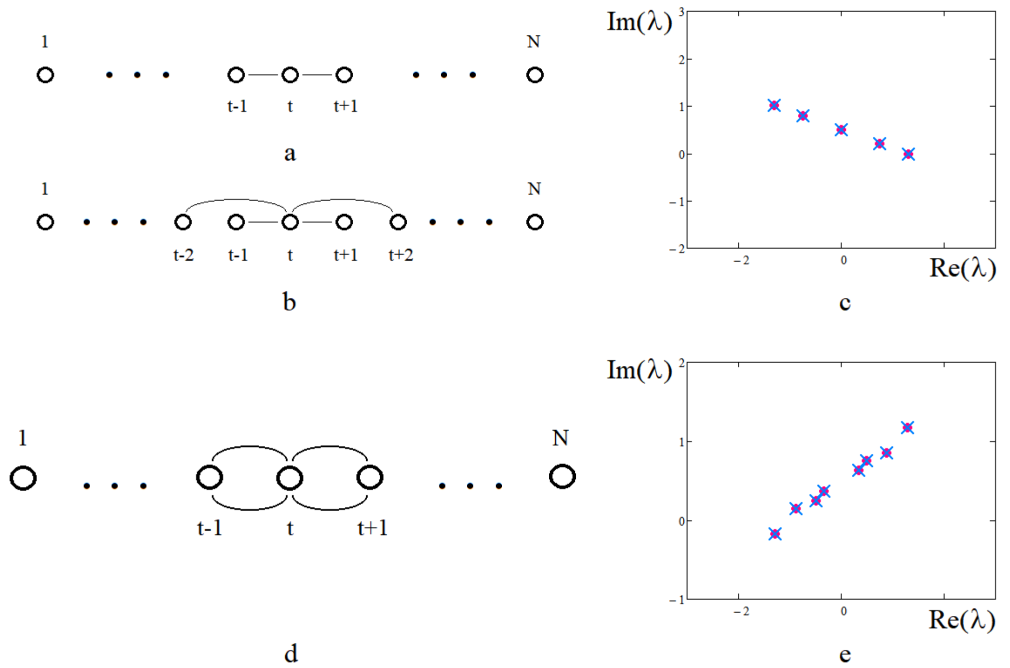


Fig. 1. One-dimensional lattices of N (a, b, d) identical DRs. The results of the numerical calculation of the eigenvalues are dots; the analytical ones are crosses (a, c), obtained for the first approximation (a): $N = 5$; $\tilde{k}_0 = 0,5$; $\kappa_{12} = 0,75-0,3i$; (c, d): $N = 4$; $\tilde{k}_0 = 0,5$; $\kappa_{12}^{\mathrm{ee}} = 0,75+0,4i$; $\kappa_{12}^{\mathrm{oo}} = -0,5-0,2i$; $\kappa_{12}^{\mathrm{eo}} = 0,25+0,1i$.(d, e) (here and below the values of the coupling coefficients are taken arbitrarily)

The solution to equation (11) has the form:

$$\theta = \theta^s = \frac{s\pi}{(N+1)}, \qquad (s = 1, 2, \dots, N). \tag{12}$$

The characteristic equation (10) together with (12) determines the N eigenvalues found in the first approximation:

$$\lambda^s = i\tilde{k}_0 + 2\kappa_{12}\cos\left[\frac{s\pi}{(N+1)}\right], \tag{13}$$

where each value λ^s corresponds s-th normalized vector (6) of natural oscillations of the resonator system:

$$b_t^s = b_0^s \sin(\theta^s t), \quad (t = 1, 2, \dots, N). \tag{14}$$

An interesting feature of the found solution is the validity of the phase distribution functions of the amplitudes of coupled oscillations obtained in the approximation of interaction of only neighboring resonators. As follows from (14), these functions also do not depend on the electromagnetic parameters, but are determined only by the number of resonators in the lattice.

Fig. 1, c shows the result of comparison of the eigenvalues of a linear lattice consisting of 5 DRs, calculated using formula (13) (crosses) and the eigenvalues obtained numerically for the truncated coupling matrix (7) (dots) for the first approximation.

The formula for the second approximation was found taking into account the coupling with the two closest resonators on each side (Fig. 1, b). For this the equation (2) was represented as:

$$\kappa_{\text{t-2,t}}\mathrm{b}_{t-2} + \kappa_{\text{t-1,t}}\mathrm{b}_{t-1} + (i\tilde{k}_0 - \lambda)\mathrm{b}_t +$$
$$+\kappa_{\text{t+1,t}}\mathrm{b}_{t+1} + \kappa_{\text{t+2,t}}\mathrm{b}_{t+2} = 0,$$

further we used the following notations: $\kappa_{t-1,t} = \kappa_{t+1,t} = \kappa_{12}$, $\kappa_{t-2,t} = \kappa_{t+2,t} = \kappa_{13}$. Then:

$$\kappa_{12}(\mathrm{b}_{t-1}+\mathrm{b}_{t+1}) + \kappa_{13}(\mathrm{b}_{t-2}+\mathrm{b}_{t+2}) + (i\tilde{k}_0-\lambda)\mathrm{b}_t = 0. \tag{15}$$

The exact analytical solution of system (15) is cumbersome, so we used perturbation theory to analytically represent the eigenvalues and eigenvectors, assuming a smallness of magnitude of the coupling coefficient $|\kappa_{13}| \ll |\kappa_{12}|$.

For this purpose, the coupling matrix K corresponding to the system of equations (15) was represented in the form:

$$K = K_1 + \kappa_{13} Q, \tag{16}$$

and used perturbation theory [23, 24], where K_1 – the coupling matrix determined for the DR system taking into account the interaction of only adjacent resonators (10), and $Q = [\![\delta_{s,t-2} + \delta_{s,t+2}]\!]$. To do this, we expanded the eigenvectors (6)

$$\mathbf{b}^s = \|\mathrm{b}_{\mathrm{t}}^s\| = \mathbf{b}_0^s + \kappa_{13}\mathbf{b}_1^s + (\kappa_{13})^2\mathbf{b}_2^s + \dots$$

and eigenvalues (2)

$$\lambda^s = \lambda_0^s + \kappa_{13}\lambda_1^s + (\kappa_{13})^2\lambda_2^s + \dots$$

in a series of powers κ_{13}. Here $\mathbf{b}_0^s$ and λ_0^s - vector (9) and eigenvalues (13) of the coupling matrix. Substituting the indicated expansions into (16) and equating the terms with the same powers, we represented, according to [24]:

$$\mathbf{b}_1^s = \sum_{j \neq s}^{N} \frac{c^{j,s}\mathbf{b}_0^j}{\lambda_0^s - \lambda_0^j},$$

where

$$c^{j,s} = \left\langle Q\mathbf{b}_0^s, \mathbf{b}_0^j \right\rangle;$$

$$\lambda_1^s = \langle Q\mathbf{b}_0^s, \mathbf{b}_0^s \rangle; \qquad (17)$$

$\langle \mathbf{a}, \ \mathbf{b} \rangle = \sum\limits_{t=1}^{N} a_t b_t^*$ is the scalar product of vectors in complex N dimensional Euclidean space.

The obtained expressions (13), (17) made it possible to estimate approximations of different orders in comparison with the results obtained numerically. Fig. 2 shows the results of calculations obtained for two lattices consisting of 10 spherical DRs, carried out for the case of fundamental oscillations H_{111} with magnetic field polarization in the center of each of the resonators, orthogonal to the lattice axis and whispering gallery modes $H_{20,20,1}$. It's obvious that the first approximation (13), found by using the tridiagonal matrix, is unsatisfactory for calculating the frequencies for the modes of main types H_{111} (Fig. 2, b). However, if the resonator field decreases rapidly with increasing distance from its surface $|\kappa_{1s}| \ll |\kappa_{12}|$ for $s > 2$, then both approximations give nearly the same result (Fig. 2, d), close to the exact values.

4 Natural oscillations of lattices with degenerate mode DRs

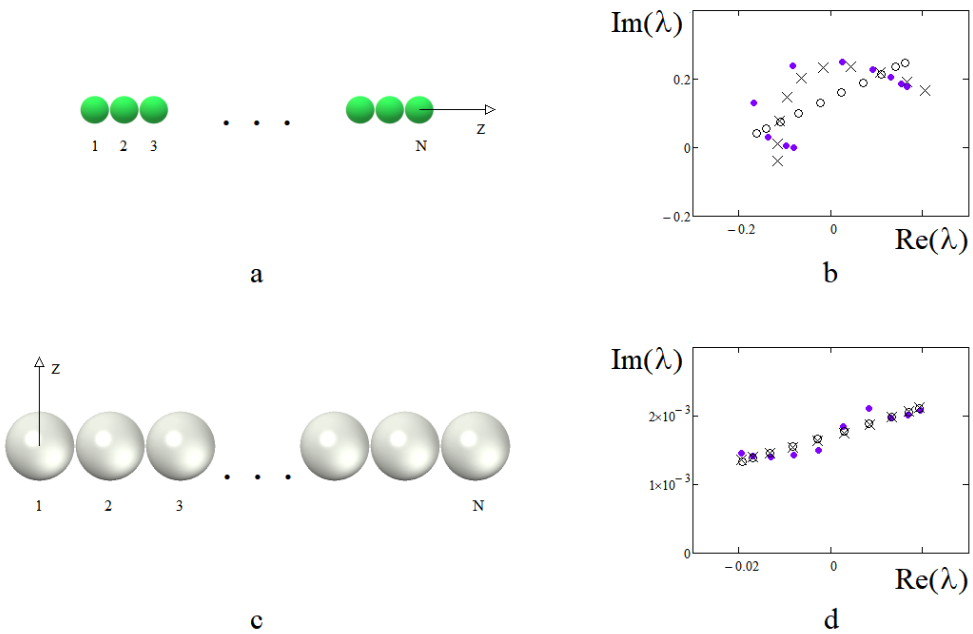
More complex solutions to the problem of natural oscillations of a lattice of identical DRs with degenerate oscillations (Fig. 3, a) were constructed based on the assumption about the behavior of their amplitudes. For simplicity, we assumed that in each resonator at one frequency there are only two orthogonal modes. We have designated these oscillations with letters: e - even and o - odd. In general $\kappa_{12}^{ee} \neq \kappa_{12}^{oo}$; in this case, the coupling coefficients of the resonators with open space are equal: $\tilde{k}_t^e = \tilde{k}_t^o = \tilde{k}_0$.

We may have formulated the conditions under which the field amplitudes of coupled oscillations of a lattice with DR degenerate modes are connected with field distributions of the same lattice with non-degenerate modes of resonators.

Indeed, in the case of a one-dimensional lattice, if a coupling matrix is Toeplitz $\kappa_{st} = \kappa_{|s-t|}$ and the distribution of the amplitudes of the natural oscillations obeys the relations:

$$\sum_{s=1}^{N} \kappa_{st} \ b_s = b_t \sum_{s=1}^{N} \kappa_s \ \alpha_s, \qquad (18)$$

where α_s do not depend on t, then the frequency spectrum of the same resonator lattice with degenerate types of natural oscillations is simply expressed through the characteristics of the same lattice of resonators with non-degenerate modes.



a b c d

Fig. 2. One-dimensional lattices of N spherical DRs with H_{111} oscillations (a) ($\varepsilon_{1r} = 9$); with $H_{20,20,1}$ oscillations (c) ($\varepsilon_{1r} = 2,2$). The first approximation results of the numerical calculation of the eigenvalues are dots; the analytical ones on (13) are cycles; for the second approximation on (16) are crosses (b, d): $N = 10$

To prove this, we represented the system of equations (2), describing the amplitudes of coupled oscillations of the DR with degenerate modes in the form of N paired equations:

$$\sum_{\mathrm{s=1}}^{\mathrm{2N}} \kappa_{\mathrm{st}}^{uv}\ \mathrm{b}_s^v + (i\tilde{k}_0 - \lambda)\mathrm{b}_t^{\mathrm{e}} = 0;$$

$$\sum_{\mathrm{s=1}}^{\mathrm{2N}} \kappa_{\mathrm{st}}^{uv}\ \mathrm{b}_s^v + (i\tilde{k}_0 - \lambda)\mathrm{b}_t^{\mathrm{o}} = 0;$$

($t = 1,\ 2, \ldots,\ N$), where uv takes on the values ee; oo; eo, depending on the relationship between modes of different types. We represented the solution of the equation system in the form:

$$\mathrm{b}_t^{\mathrm{e,o}} = A^{\mathrm{e,o}}\mathrm{b}_t, \tag{19}$$

where amplitudes $A^{\mathrm{e,o}}$ do not depend on the DR number, and b_t are the solution of the system of equations (2) for DR with non-degenerate oscillations. Substituting (19) into (18), we obtain a system of two linear equations for unknown amplitudes $A^{\mathrm{e,o}}$:

$$\begin{cases} A^{\mathrm{e}} \sum\limits_{\mathrm{s=1}}^{\mathrm{N}} \kappa_{\mathrm{st}}^{\mathrm{ee}}\ \mathrm{b}_s + A^{\mathrm{o}} \sum\limits_{\mathrm{s=1}}^{\mathrm{N}} \kappa_{\mathrm{st}}^{\mathrm{eo}}\ \mathrm{b}_s + (i\tilde{k}_0 - \lambda)A^{\mathrm{e}}\mathrm{b}_t = 0; \\ A^{\mathrm{o}} \sum\limits_{\mathrm{s=1}}^{\mathrm{N}} \kappa_{\mathrm{st}}^{\mathrm{oo}}\ \mathrm{b}_s + A^{\mathrm{e}} \sum\limits_{\mathrm{s=1}}^{\mathrm{N}} \kappa_{\mathrm{st}}^{\mathrm{eo}}\ \mathrm{b}_s + (i\tilde{k}_0 - \lambda)A^{\mathrm{o}}\mathrm{b}_t = 0 \end{cases} \tag{20}$$

or taking into account (17)

$$\begin{cases} \left[\sum\limits_{\mathrm{s=1}}^{\mathrm{N}} \kappa_{\mathrm{s}}^{\mathrm{ee}}\ \alpha_s + (i\tilde{k}_0 - \lambda)\right] \cdot A^{\mathrm{e}} + \left[\sum\limits_{\mathrm{s=1}}^{\mathrm{N}} \kappa_{\mathrm{s}}^{\mathrm{eo}}\ \alpha_s\right] \cdot A^{\mathrm{o}} = 0; \\ \left[\sum\limits_{\mathrm{s=1}}^{\mathrm{N}} \kappa_{\mathrm{s}}^{\mathrm{eo}}\ \alpha_s\right] \cdot A^{\mathrm{e}} + \left[\sum\limits_{\mathrm{s=1}}^{\mathrm{N}} \kappa_{\mathrm{s}}^{\mathrm{oo}}\ \alpha_s + (i\tilde{k}_0 - \lambda)\right] \cdot A^{\mathrm{o}} = 0. \end{cases} \tag{21}$$

A non-trivial solution to the system determines the eigenvalues

$$\lambda^{\pm} = i\tilde{k}_0 + \frac{1}{2}\left[\sum_{\mathrm{s=1}}^{\mathrm{N}} (\kappa_{\mathrm{s}}^{\mathrm{ee}} + \kappa_{\mathrm{s}}^{\mathrm{oo}})\ \alpha_s \pm d\right]; \tag{22}$$

$$d = \left\{\left[\sum_{\mathrm{s=1}}^{\mathrm{N}} (\kappa_{\mathrm{s}}^{\mathrm{ee}} + \kappa_{\mathrm{s}}^{\mathrm{oo}})\ \alpha_s\right]^2 - 4\left[\left(\sum_{\mathrm{s=1}}^{\mathrm{N}} \kappa_{\mathrm{s}}^{\mathrm{ee}}\alpha_s\right)\left(\sum_{\mathrm{s=1}}^{\mathrm{N}} \kappa_{\mathrm{s}}^{\mathrm{oo}}\alpha_s\right) - \left(\sum_{\mathrm{s=1}}^{\mathrm{N}} \kappa_{\mathrm{s}}^{\mathrm{eo}}\alpha_s\right)^2\right]\right\}^{1/2};$$

and the ratio of amplitudes:

$$\frac{A^{\mathrm{o}}}{A^{\mathrm{e}}} = \frac{-\left[\sum\limits_{\mathrm{s=1}}^{\mathrm{N}} \kappa_{\mathrm{s}}^{\mathrm{ee}}\ \alpha_s + (i\tilde{k}_0 - \lambda)\right]}{\sum\limits_{\mathrm{s=1}}^{\mathrm{N}} \left[\kappa_{\mathrm{s}}^{\mathrm{eo}}\alpha_s\right]}. \tag{23}$$

It also follows from the obtained results that in the absence of a coupling between the degenerate modes of all resonators ($\kappa_s^{\mathrm{eo}} = 0$), the natural oscillations disintegrate into oscillations of two set of separate submodes, the frequencies of which are determined by the natural values found from two independent equations.

5 Natural oscillations of one-dimensional lattices with degenerate mode DRs

As an example, coupled oscillations of a one-dimensional DR lattice were considered. For simplicity, only the first approximation was found, i.e. it was assumed that the resonators are coupled only with neighboring ones (Fig. 1, d), while the coupling coefficients are non-zero: $\kappa_{t-1,t}^{\mathrm{e,e}} = \kappa_{t,t+1}^{\mathrm{e,e}} = \kappa_{12}^{\mathrm{e,e}}$; $\kappa_{t-1,t}^{\mathrm{o,o}} = \kappa_{t,t+1}^{\mathrm{o,o}} = \kappa_{12}^{\mathrm{o,o}}$; $\kappa_{t-1,t}^{\mathrm{e,o}} = \kappa_{t,t+1}^{\mathrm{e,o}} = \kappa_{12}^{\mathrm{e,o}} = \kappa_{12}^{\mathrm{o,e}}$, and the degenerate oscillations in each DR are orthogonal: $\kappa_{tt}^{\mathrm{eo}} = 0$.

In this case, equations (18) for the t-th DR lattice with non-zero terms are represented in the form:

$$\begin{cases} \kappa_{12}^{\mathrm{ee}}(\mathrm{b}_{t-1}^{\mathrm{e}} + \mathrm{b}_{t+1}^{\mathrm{e}}) + \kappa_{12}^{\mathrm{eo}}(\mathrm{b}_{t-1}^{\mathrm{o}} + \mathrm{b}_{t+1}^{\mathrm{o}}) + (i\tilde{k}_0 - \lambda)\mathrm{b}_t^{\mathrm{e}} = 0 \\ \kappa_{12}^{\mathrm{eo}}(\mathrm{b}_{t-1}^{\mathrm{e}} + \mathrm{b}_{t+1}^{\mathrm{e}}) + \kappa_{12}^{\mathrm{oo}}(\mathrm{b}_{t-1}^{\mathrm{o}} + \mathrm{b}_{t+1}^{\mathrm{o}}) + (i\tilde{k}_0 - \lambda)\mathrm{b}_t^{\mathrm{o}} = 0 \end{cases}; \quad (t = 1, 2, \ldots, N) \tag{24}$$

The solution to system (24) was sought in the form:

$$\mathrm{b}_t^{\mathrm{e}} = A^{\mathrm{e}}\sin(\theta t), \quad \mathrm{b}_t^{\mathrm{o}} = A^{\mathrm{o}}\sin(\theta t), \tag{25}$$

where also

$$\theta = \theta^s = \frac{s\pi}{(N+1)}; \quad (s = 1, 2, \ldots, N). \tag{26}$$

Condition (17), taking into account (25), takes the form:

$$\sum_{\mathrm{s=1}}^{\mathrm{N}} \kappa_{\mathrm{st}}\ \mathrm{b}_s \to \kappa_{\mathrm{t\text{-}1,t}}\mathrm{b}_{t-1} + \kappa_{\mathrm{t+1,t}}\mathrm{b}_{t+1} =$$

$$= \kappa_{12}(\mathrm{b}_{t-1} + \mathrm{b}_{t+1}) = 2\mathrm{b}_t\kappa_{12}\cos(\theta),$$

where do we find the value: $\alpha_s = 2\cos(\theta)$. From (22), (23) we determined the relative amplitudes:

$$A^{\mathrm{o}}/A^{\mathrm{e}} = -[2\cos(\theta)\kappa_{12}^{\mathrm{ee}} + (i\tilde{k}_0 - \lambda)]/\left[2\cos(\theta)\kappa_{12}^{\mathrm{eo}}\right] \tag{27}$$

as well as natural oscillation frequencies:

$$\lambda^{\pm} = i\tilde{k}_0 + \cos(\theta)\left[(\kappa_{12}^{\mathrm{ee}} + \kappa_{12}^{\mathrm{oo}}) \pm d\right], \tag{28}$$

where

$$d = \left\{(\kappa_{12}^{\mathrm{ee}} + \kappa_{12}^{\mathrm{oo}})^2 - 4\left[\kappa_{12}^{\mathrm{ee}}\kappa_{12}^{\mathrm{oo}} - (\kappa_{12}^{\mathrm{eo}})^2\right]\right\}^{1/2}. \tag{29}$$

The correctness of the reasoning carried out is confirmed by a comparison of the eigenvalues obtained from (26), (28) (crosses) and a numerical calculation of the eigenvalues of the tridiagonal coupling matrix (dots) (Fig. 1, e).

6 Natural oscillations of two-dimensional lattices DR

In a similar way, the natural oscillations of a two-dimensional rectangular lattice consisting of $N \times M$ identical resonators were calculated (Fig. 3, a). In this case "coordinates" of each resonator were determined by two numbers (s, t), s denoted the horizontal position number, and t the vertical position number of the DR in a rectangular lattice. When solving the system of equations (2) for the s, t-th DR, the first approximation was considered to be taking into account the coupling of each of the resonators only with the nearest ones of the lattice (Fig. 3, a):

$$\kappa_{\text{s-1,t-1}|\text{s,t}}\text{b}_{\text{s-1,t-1}} + \kappa_{\text{s-1,t}|\text{s,t}}\text{b}_{\text{s-1,t}} + \kappa_{\text{s-1,t+1}|\text{s,t}}\text{b}_{\text{s-1,t+1}} + \\ +\kappa_{\text{s,t-1}|\text{s,t}}\text{b}_{\text{s,t-1}} + (i\tilde{k}_0 - \lambda)\text{b}_{s,t} + \kappa_{\text{s,t+1}|\text{s,t}}\text{b}_{\text{s,t+1}} + \\ +\kappa_{\text{s+1,t-1}|\text{s,t}}\text{b}_{\text{s+1,t-1}} + \kappa_{\text{s+1,t}|\text{s,t}}\text{b}_{\text{s+1,t}} + \\ +\kappa_{\text{s+1,t+1}|\text{s,t}}\text{b}_{\text{s+1,t+1}} = 0,$$

$(s = 1, 2, \dots, N;\ t = 1, 2, \dots, M)$. Here $\kappa_{u,v|s,t}$ – mutual coupling coefficients between resonator u, v and the resonator s, t.

For identical DRs, the coupling coefficients were designated as $\kappa_{s-1,t-1|s,t} = \kappa_{s-1,t+1|s,t} = \kappa_{s+1,t-1|s,t} = \kappa_{s+1,t+1|s,t} = \kappa_{xy}$; $\kappa_{s-1,t|s,t} = \kappa_{s+1,t|s,t} = \kappa_x$; $\kappa_{s,t-1|s,t} = \kappa_{s,t+1|s,t} = \kappa_y$. In this case, the system of equations takes a more compact form:

$$\kappa_x(\text{b}_{\text{s-1,t}} + \text{b}_{s+1,t}) + \kappa_y(\text{b}_{\text{s,t-1}} + \text{b}_{s,t+1}) + (i\tilde{k}_0 - \lambda)\text{b}_{s,t} + \\ + \kappa_{xy}(\text{b}_{\text{s-1,t-1}} + \text{b}_{\text{s-1,t+1}} + \text{b}_{\text{s+1,t-1}} + \text{b}_{\text{s+1,t+1}}) = 0. \tag{30}$$

The solution to the system of equations (30) was sought in the form:

$$\text{b}_{s,t} = \text{b}_0 \sin(\theta_x s)\sin(\theta_y t), \tag{31}$$

where it was assumed that b_0 also does not depend on the resonator number.

Substituting (31) into (30), we again found the characteristic equation, which is independent of the DR indices:

$$2\kappa_{\text{x}}\cos(\theta_x) + 2\kappa_{\text{y}}\cos(\theta_y) + (i\tilde{k}_0 - \lambda) + \\ + 4\kappa_{xy}\cos(\theta_x)\cos(\theta_y) = 0. \tag{32}$$

Equation (32) was supplemented by the conditions of symmetry of natural oscillations: $|\text{b}_{u,t}| = |\text{b}_{N-u+1,t}|$ and $|\text{b}_{s,v}| = |\text{b}_{s,M-v+1}|$, where taking into account (31)

$$|\sin(\theta_x u)| = |\sin[\theta_x(N - u + 1)]|\,; \\ |\sin(\theta_y v)| = |\sin[\theta_y(M - v + 1)]|\,. \tag{33}$$

The solution of equations (33), as well as (11), (12), we know:

$$\theta_x = \theta_x^n = \frac{n\pi}{(N+1)};\quad \theta_y = \theta_y^m = \frac{m\pi}{(M+1)}; \\ (n = 1, 2, \dots, N;\ m = 1, 2, \dots, M). \tag{34}$$

Expressions (34) together with (32) determine the eigenvalues of a two-dimensional lattice of $N \times M$ identical DRs in the first approximation:

$$\lambda^{n,m} = i\tilde{k}_0 + 2\kappa_{\text{x}}\cos\left[\frac{n\pi}{(N+1)}\right] + 2\kappa_{\text{y}}\cos\left[\frac{m\pi}{(M+1)}\right] + \\ + 4\kappa_{xy}\cos\left[\frac{n\pi}{(N+1)}\right]\cos\left[\frac{m\pi}{(M+1)}\right]. \tag{35}$$

The result of the numerical (points) and analytical (35) (crosses) calculation of the eigenvalues of the coupling operator (4) for the rectangular $N \times M = 4 \times 4$ DR lattice is shown in Fig. 3, b.

The second approximation for a two-dimensional lattice in analytical form can be obtained similarly to the approximation obtained for a one-dimensional lattice using perturbation theory.

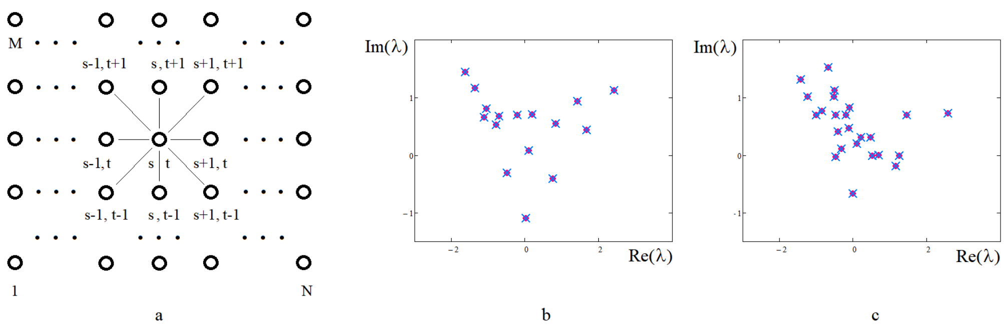


Fig. 3. Two-dimensional (a) lattice of identical resonators. The results of the numerical calculation of the eigenvalues are dots; the analytical ones are crosses; b: $N = 4$; $M = 4$; $\tilde{k}_0 = 0,5$; $\kappa_x = 0,75 - 0,3i$; $\kappa_y = 0,5 + 0,2i$; $\kappa_{xy} = 0,15 + 0,3i$; c: $N = 4$; $M = 3$; $\tilde{k}_0 = 0,5$; $\kappa_x^{\text{ee}} = 0,75 - 0,3i$; $\kappa_x^{\text{oo}} = -0,3 + 0,3i$; $\kappa_y^{\text{ee}} = 0,5 + 0,2i$; $\kappa_y^{\text{oo}} = -0,25 + 0,2i$; $\kappa_{xy}^{\text{ee}} = 0,2 + 0,2i$; $\kappa_{xy}^{\text{oo}} = 0,15 + 0,1i$; $\kappa_x^{\text{eo}} = 0,15 - 0,1i$; $\kappa_y^{\text{eo}} = 0,2 - 0,15i$; $\kappa_{xy}^{\text{eo}} = 0,1 + 0,1i$

7 Natural oscillations of two-dimensional lattices with degenerate mode DRs

In the case of a two-dimensional rectangular lattice of $N \times M$ identical resonators with two degenerate types of natural oscillations (Fig. 3, a), the designations of the coupling coefficients of the degenerate oscillations were retained, the same as in 6: $\tilde{k}_s^{\mathrm{e}} = \tilde{k}_s^{\mathrm{o}} = \tilde{k}_0$; $\kappa^{uv}_{s-1,t-1|s,t} = \kappa^{uv}_{s-1,t+1|s,t} = \kappa^{uv}_{s+1,t-1|s,t} = \kappa^{uv}_{s+1,t+1|s,t} = \kappa^{uv}_{xy}$; $\kappa^{uv}_{s-1,t|s,t} = \kappa^{uv}_{s+1,t|s,t} = \kappa^{uv}_{x}$; $\kappa^{uv}_{s,t-1|s,t} = \kappa^{uv}_{s,t+1|s,t} = \kappa^{uv}_{y}$, where u,v take values e or o depending on the mode parity of the partial resonators, and the degenerate modes of each DR are orthogonal to each other: $\kappa^{\mathrm{eo}}_{tt} = 0$. Here also $\kappa^{\mathrm{eo}}_{u,v|s,t}$ – mutual coupling coefficients between resonator u,v and the resonator s,t. The “coordinates” of each resonator we also defined by two numbers: (s,t), where s denoted the horizontal position number, and the t vertical position number of the DR in a rectangular lattice. When solving the system of equations (2) in the first approximation for the s,t-th DR, the coupling of each of the resonators was taken into account only with the nearest resonators of the lattice, using above designations:

$$\begin{gathered}
\kappa_x^{\mathrm{ee}}(\mathrm{b}^{\mathrm{e}}_{\text{s-1,t}}+\mathrm{b}^{\mathrm{e}}_{\text{s+1,t}})+\kappa_y^{\mathrm{ee}}(\mathrm{b}^{\mathrm{e}}_{\text{s,t-1}}+\mathrm{b}^{\mathrm{e}}_{\text{s,t+1}})+(i\tilde{k}_0-\lambda)\mathrm{b}^{\mathrm{e}}_{s,t}+\\
+\kappa_{xy}^{\mathrm{ee}}(\mathrm{b}^{\mathrm{e}}_{\text{s-1,t-1}}+\mathrm{b}^{\mathrm{e}}_{\text{s-1,t+1}}+\mathrm{b}^{\mathrm{e}}_{\text{s+1,t-1}}+\mathrm{b}^{\mathrm{e}}_{\text{s+1,t+1}})+\\
+\kappa_x^{\mathrm{eo}}(\mathrm{b}^{\mathrm{o}}_{\text{s-1,t}}+\mathrm{b}^{\mathrm{o}}_{\text{s+1,t}})+\kappa_y^{\mathrm{eo}}(\mathrm{b}^{\mathrm{o}}_{\text{s,t-1}}+\mathrm{b}^{\mathrm{o}}_{\text{s,t+1}})+\\
\kappa_{xy}^{\mathrm{eo}}(\mathrm{b}^{\mathrm{o}}_{\text{s-1,t-1}}+\mathrm{b}^{\mathrm{o}}_{\text{s-1,t+1}}+\mathrm{b}^{\mathrm{o}}_{\text{s+1,t-1}}+\mathrm{b}^{\mathrm{o}}_{\text{s+1,t+1}})=0,\\
\kappa_x^{\mathrm{oo}}(\mathrm{b}^{\mathrm{o}}_{\text{s-1,t}}+\mathrm{b}^{\mathrm{o}}_{\text{s+1,t}})+\kappa_y^{\mathrm{oo}}(\mathrm{b}^{\mathrm{o}}_{\text{s,t-1}}+\mathrm{b}^{\mathrm{o}}_{\text{s,t+1}})+(i\tilde{k}_0-\lambda)\mathrm{b}^{\mathrm{o}}_{s,t}+\\
+\kappa_{xy}^{\mathrm{oo}}(\mathrm{b}^{\mathrm{o}}_{\text{s-1,t-1}}+\mathrm{b}^{\mathrm{o}}_{\text{s-1,t+1}}+\mathrm{b}^{\mathrm{o}}_{\text{s+1,t-1}}+\mathrm{b}^{\mathrm{o}}_{\text{s+1,t+1}})+\\
+\kappa_x^{\mathrm{eo}}(\mathrm{b}^{\mathrm{e}}_{\text{s-1,t}}+\mathrm{b}^{\mathrm{e}}_{\text{s+1,t}})+\kappa_y^{\mathrm{eo}}(\mathrm{b}^{\mathrm{e}}_{\text{s,t-1}}+\mathrm{b}^{\mathrm{e}}_{\text{s,t+1}})+\\
+\kappa_{xy}^{\mathrm{eo}}(\mathrm{b}^{\mathrm{e}}_{\text{s-1,t-1}}+\mathrm{b}^{\mathrm{e}}_{\text{s-1,t+1}}+\mathrm{b}^{\mathrm{e}}_{\text{s+1,t-1}}+\mathrm{b}^{\mathrm{e}}_{\text{s+1,t+1}})=0\\
(s=1,2,\ldots,N;\quad t=1,2,\ldots,M).
\end{gathered} \tag{36}$$

The solution to the system of equations (36) was sought in the form of standing waves of the Bloch type:

$$\mathrm{b}^{\mathrm{e,o}}_{s,t}=\mathrm{b}^{\mathrm{e,o}}_0\sin(\theta_x s)\sin(\theta_y t), \tag{37}$$

where it was also assumed that $\mathrm{b}_0^{e,o}$ does not depend on the number of resonators.

Substituting (37) into (36), we again obtained a system of equations, that does not depend on the DR number:

$$\begin{gathered}
\{2[\kappa_{\mathrm{x}}^{\mathrm{ee}}\cos(\theta_x)+\kappa_{\mathrm{y}}^{\mathrm{ee}}\cos(\theta_y)+2\kappa_{xy}^{\mathrm{ee}}\cos(\theta_x)\cos(\theta_y)]+\\
+(i\tilde{k}_0-\lambda)\}\cdot b_0^{\mathrm{e}}+2[\kappa_{\mathrm{x}}^{\mathrm{eo}}\cos(\theta_x)+\kappa_{\mathrm{y}}^{\mathrm{eo}}\cos(\theta_y)+\\
+2\kappa_{xy}^{\mathrm{eo}}\cos(\theta_x)\cos(\theta_y)]\cdot b_0^{\mathrm{o}}=0;\\
2[\kappa_{\mathrm{x}}^{\mathrm{eo}}\cos(\theta_x)+\kappa_{\mathrm{y}}^{\mathrm{eo}}\cos(\theta_y)+2\kappa_{xy}^{\mathrm{eo}}\cos(\theta_x)\cos(\theta_y)]\cdot b_0^{\mathrm{e}}+\\
+\{2[\kappa_{\mathrm{x}}^{\mathrm{oo}}\cos(\theta_x)+\kappa_{\mathrm{y}}^{\mathrm{oo}}\cos(\theta_y)+2\kappa_{xy}^{\mathrm{oo}}\cos(\theta_x)\cos(\theta_y)]+\\
+(i\tilde{k}_0-\lambda)\}\cdot b_0^{\mathrm{o}}=0.
\end{gathered} \tag{38}$$

The non-trivial solution (38) determines the eigenvalues:

$$\begin{gathered}
\lambda^{\pm}=i\tilde{k}_0+(\kappa_x^{\mathrm{ee}}+\kappa_x^{\mathrm{oo}})\cos(\theta_x)+(\kappa_y^{\mathrm{ee}}+\kappa_y^{\mathrm{oo}})\cos(\theta_y)+\\
+2(\kappa_{xy}^{\mathrm{ee}}+\kappa_{xy}^{\mathrm{oo}})\cos(\theta_x)\cos(\theta_y)\pm d,
\end{gathered} \tag{39}$$

where

$$\begin{gathered}
d=\{[(\kappa_x^{\mathrm{ee}}+\kappa_x^{\mathrm{oo}})\cos(\theta_x)+(\kappa_y^{\mathrm{ee}}+\kappa_y^{\mathrm{oo}})\cos(\theta_y)+\\
+2(\kappa_{xy}^{\mathrm{ee}}+\kappa_{xy}^{\mathrm{oo}})\cos(\theta_x)\cos(\theta_y)]^2-\\
-4[\kappa_x^{\mathrm{ee}}\cos(\theta_x)+\kappa_y^{\mathrm{ee}}\cos(\theta_y)+2\kappa_{xy}^{\mathrm{ee}}\cos(\theta_x)\cos(\theta_y)]\times\\
\times[\kappa_x^{\mathrm{oo}}\cos(\theta_x)+\kappa_y^{\mathrm{oo}}\cos(\theta_y)+2\kappa_{xy}^{\mathrm{oo}}\cos(\theta_x)\cos(\theta_y)]+\\
+4[\kappa_x^{\mathrm{eo}}\cos(\theta_x)+\kappa_y^{\mathrm{eo}}\cos(\theta_y)+2\kappa_{xy}^{\mathrm{eo}}\cos(\theta_x)\cos(\theta_y)]^2\Big\}^{\frac{1}{2}}.
\end{gathered} \tag{40}$$

Equation (39) was supplemented by symmetry conditions: $\left|\mathrm{b}^{\mathrm{e,o}}_{u,t}\right| = \left|\mathrm{b}^{\mathrm{e,o}}_{N-u+1,t}\right|$ and $\left|\mathrm{b}^{\mathrm{e,o}}_{s,v}\right| = \left|\mathrm{b}^{\mathrm{e,o}}_{s,M-v+1}\right|$, from where, taking into account (37), we found:

$$\begin{gathered}
\theta_x=\frac{n\pi}{(N+1)};\quad \theta_y=\frac{m\pi}{(M+1)};\\
(n=1,2,\ldots,N;\quad m=1,2,\ldots,M).
\end{gathered} \tag{41}$$

The expressions (41) together with (39), (40) in general define the $2\times(N\times M)$ eigenvalues obtained in the first approximation of a two-dimensional lattice of identical DR with two degenerate types of natural oscillations.

The result of the numerical (points) and analytical (37)-(39) calculation of the eigenvalues of the coupling operator (4) for a rectangular $N\times M=4\times 3$ DR lattice is shown in Fig. 3, c.

8 Natural oscillations of three-dimensional lattices DR

Three-dimensional lattices usually contain a very large number of resonators, which significantly complicates the calculation of their parameters, so obtaining analytical relationships for them is a very urgent task. We considered the parameters of coupled oscillations of a three-dimensional rectangular lattice of identical $N\times M\times L$ resonators (Fig. 4, a). In this case the coordinates of each resonator of the lattice were determined by three indices (s,t,u), where s denotes the number of the horizontal position along the x axis, t the number of the horizontal position along the y axis and l the number of the vertical position of the DR along the z axis. It was assumed also that the mutual coupling coefficients of the resonators are symmetrical:

$$\begin{aligned}
&\kappa_{s-1,t,l|s,t,l} = \kappa_{s+1,t,l|s,t,l} = \kappa_x;\\
&\kappa_{s,t-1,l|s,t,l} = \kappa_{s,t+1,l|s,t,l} = \kappa_y;\\
&\kappa_{s,t,l-1|s,t,l} = \kappa_{s,t,l+1|s,t,l} = \kappa_z;\\
&\kappa_{s-1,t-1,l|s,t,l} = \kappa_{s-1,t+1,l|s,t,l} = \kappa_{s+1,t-1,l|s,t,l} =\\
&= \kappa_{s+1,t+1,l|s,t,l} = \kappa_{xy};\\
&\kappa_{s-1,t,l-1|s,t,l} = \kappa_{s-1,t,l+1|s,t,l} = \kappa_{s+1,t,l-1|s,t,l} =\\
&= \kappa_{s+1,t,l+1|s,t,l} = \kappa_{xz};\\
&\kappa_{s,t-1,l-1|s,t,l} = \kappa_{s,t-1,l+1|s,t,l} = \kappa_{s,t+1,l-1|s,t,l} =\\
&= \kappa_{s,t+1,l+1|s,t,l} = \kappa_{yz};\\
&\kappa_{s-1,t-1,l-1|s,t,l} = \kappa_{s+1,t+1,l+1|s,t,l} =\\
&= \kappa_{s+1,t-1,l-1|s,t,l} = \kappa_{s-1,t+1,l+1|s,t,l} = \kappa_{xyz};\\
&\kappa_{s-1,t+1,l-1|s,t,l} = \kappa_{s+1,t-1,l+1|s,t,l} =\\
&= \kappa_{s+1,t+1,l-1|s,t,l} = \kappa_{s-1,t-1,l+1|s,t,l} = \kappa_{xyz}.
\end{aligned} \tag{42}$$

Then the system of equations (2) for the DR with "coordinates" (s, t, l), in the first approximation, taking into account the coupling only with the nearest resonators of the lattice, can be represented in the form:

$$\begin{aligned}
&\kappa_x(\mathrm{b_{s\text{-}1,t,l}} + \mathrm{b_{s+1,t,l}}) + \kappa_y(\mathrm{b_{s,t\text{-}1,l}} + \mathrm{b_{s,t+1,l}}) +\\
&+ \kappa_{\mathrm{z}}(\mathrm{b_{s,t,l\text{-}1}} + \mathrm{b_{s,t,l+1}}) + (i\tilde{k}_0 - \lambda)\mathrm{b}_{s,t,l} +\\
&+ \kappa_{\mathrm{xy}}(\mathrm{b_{s\text{-}1,t\text{-}1,u}} + \mathrm{b_{s+1,t+1,l}} + \mathrm{b_{s+1,t\text{-}1,l}} + \mathrm{b_{s\text{-}1,t+1,l}}) +\\
&+ \kappa_{xz}(\mathrm{b_{s\text{-}1,t,l\text{-}1}} + \mathrm{b_{s+1,t,l\text{-}1}} + \mathrm{b_{s\text{-}1,t,l+1}} + \mathrm{b_{s+1,t,l+1}}) +\\
&+ \kappa_{yz}(\mathrm{b_{s,t\text{-}1,l\text{-}1}} + \mathrm{b_{s,t+1,l\text{-}1}} + \mathrm{b_{s,t\text{-}1,l+1}} + \mathrm{b_{s,t+1,l+1}}) +\\
&+ \kappa_{xyz}(\mathrm{b_{s\text{-}1,t\text{-}1,l\text{-}1}} + \mathrm{b_{s+1,t+1,l+1}} + \mathrm{b_{s+1,t\text{-}1,l\text{-}1}} +\\
&+ \mathrm{b_{s\text{-}1,t+1,l+1}} + \mathrm{b_{s\text{-}1,t+1,l\text{-}1}} + \mathrm{b_{s+1,t\text{-}1,l+1}} +\\
&+ \mathrm{b_{s+1,t+1,l\text{-}1}} + \mathrm{b_{s\text{-}1,t\text{-}1,l+1}}) = 0,\\
&\qquad (s = 1, 2, \ldots, N;\ t = 1, 2, \ldots, M;\ l = 1, 2, \ldots, L).
\end{aligned} \tag{43}$$

The solution of system (43) was again represented in the form:

$$\mathrm{b}_{s,t,l} = \mathrm{b}_0 \sin(\theta_x s) \sin(\theta_y t) \sin(\theta_z l). \tag{44}$$

Substituting (44) into (43), we found, taking into account (42), after simple transformations:

$$\begin{gathered}
2\kappa_{\mathrm{x}} \cos(\theta_x) + 2\kappa_{\mathrm{y}} \cos(\theta_y) + 2\kappa_z \cos(\theta_z) + (i\tilde{k}_0 - \lambda) +\\
+4\kappa_{xy} \cos(\theta_x)\cos(\theta_y) + 4\kappa_{xz} \cos(\theta_x)\cos(\theta_z) +\\
+4\kappa_{yz}\cos(\theta_y)\cos(\theta_z) + 8\kappa_{xyz}\cos(\theta_x)\cos(\theta_y)\cos(\theta_z) = 0,
\end{gathered}$$

from where:

$$\begin{aligned}
\lambda = {}& i\tilde{k}_0 + 2\kappa_{\mathrm{x}} \cos(\theta_x) + 2\kappa_{\mathrm{y}} \cos(\theta_y) + 2\kappa_z \cos(\theta_z) +\\
&+ 4\kappa_{xy} \cos(\theta_x)\cos(\theta_y) + 4\kappa_{xz} \cos(\theta_x)\cos(\theta_z) +\\
&+ 4\kappa_{yz} \cos(\theta_y)\cos(\theta_z) + 8\kappa_{xyz} \cos(\theta_x)\cos(\theta_y)\cos(\theta_z).
\end{aligned} \tag{45}$$

Also using the symmetry properties of the fields of natural oscillations of a rectangular lattice: $|\mathrm{b}_{r,t,u}| = |\mathrm{b}_{N-r+1,t,u}|$; $|\mathrm{b}_{s,v,u}| = |\mathrm{b}_{s,M-v+1,u}|$; $|\mathrm{b}_{s,t,w}| = |\mathrm{b}_{s,t,L-w+1}|$, we similarly determined:

$$\begin{gathered}
\theta_x = \theta_x^n = \frac{n\pi}{(N+1)};\ \theta_y = \theta_y^m = \frac{m\pi}{(M+1)};\\
\theta_z = \theta_z^l = \frac{l\pi}{(L+1)},\\
(n = 1, 2, \ldots, N;\ m = 1, 2, \ldots, M;\ l = 1, 2, \ldots, L);
\end{gathered} \tag{46}$$

$$\begin{aligned}
\lambda^{n,m,l} = {}& i\tilde{k}_0 + 2\kappa_{\mathrm{x}} \cos\left[\frac{n\pi}{(N+1)}\right] + 2\kappa_{\mathrm{y}} \cos\left[\frac{m\pi}{(M+1)}\right] +\\
&+ 2\kappa_z \cos\left[\frac{l\pi}{(L+1)}\right] +\\
&+ 4\kappa_{xy} \cos\left[\frac{n\pi}{(N+1)}\right] \cos\left[\frac{m\pi}{(M+1)}\right] +\\
&+ 4\kappa_{xz} \cos\left[\frac{n\pi}{(N+1)}\right] \cos\left[\frac{l\pi}{(L+1)}\right] +\\
&+ 4\kappa_{yz} \cos\left[\frac{m\pi}{(M+1)}\right] \cos\left[\frac{l\pi}{(L+1)}\right] +\\
&+ 8\kappa_{xyz} \cos\left[\frac{n\pi}{(N+1)}\right] \cos\left[\frac{m\pi}{(M+1)}\right] \cos\left[\frac{l\pi}{(L+1)}\right].
\end{aligned} \tag{47}$$

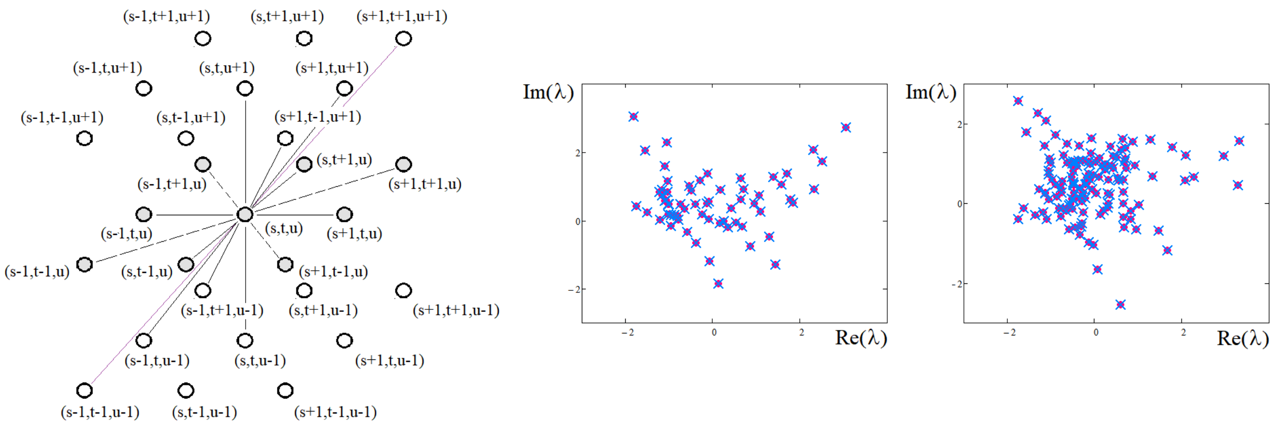
In particular, as follows from (35), (47), (3), if the coupling coefficients of the DR in different "directions" of a square lattice are equal, then the natural oscillations degenerate: $\lambda^{n,m} = \lambda^{m,n}$ for a two-dimensional and similarly $\lambda^{n,m,l} = \lambda^{m,n,l}$ for a three-dimensional lattice.

Real amplitudes (9), obtained in the approximation of taking into account the coupling only between neighboring resonators, gives rise to another type of degenerate oscillations of the system. Indeed, if in the case of a one-dimensional lattice the solutions of the system of equations are represented in the form:

$$\mathrm{b}_n = \mathrm{b}_0 \cos(\theta n), \quad (n = 1, 2, \ldots, N)$$

then we will get the same set of frequencies of coupled oscillations (13), as when choosing the amplitude distribution (9). In this case, the symmetry conditions $|b_1| = |b_N|$ will also be met. A similar statement is obviously also true for the two-dimensional and three-dimensional lattices considered above. In this case, the degree of degeneration increases in proportion to the number of different combinations of the specified distributions in different directions.

Fig. 4, b shows the results of comparison of the eigenvalues obtained from (2) (crosses) and numerical calculations (dots) for the truncated coupling matrix of a rectangular lattice $N \times M \times L = 4 \times 4 \times 4$.



a b c

Fig. 4. Three-dimensional lattice (a) of identical resonators. Results of numerical calculation – points; results of analytical calculation in the first approximation – crosses (b): $N = 4$; $M = 4$; $L = 4$; $\tilde{k}_0 = 0,5$; $\kappa_x = 0,6 - 0,3i$; $\kappa_y = 0,5 + 0,2i$; $\kappa_z = 0,4 + 0,2i$; $\kappa_{xy} = 0,15 + 0,3i$; $\kappa_{xz} = 0,1 - 0,2i$; $\kappa_{yz} = 0,15 + 0,2i$; $\kappa_{xyz} = -0,1 + 0,3i$; (c): $\kappa_x^{\mathrm{e}} = 0,6 - 0,3i$; $\kappa_x^{\mathrm{o}} = 0,3 + 0,15i$; $\kappa_x^{\mathrm{eo}} = 0,2 + 0,1i$; $\kappa_y^{\mathrm{e}} = 0,5 + 0,2i$; $\kappa_y^{\mathrm{o}} = 0,2 + 0,1i$; $\kappa_y^{\mathrm{eo}} = 0,1 + 0,1i$; $\kappa_z^{\mathrm{e}} = 0,4 + 0,2i$; $\kappa_z^{\mathrm{o}} = 0,2 - 0,1i$; $\kappa_z^{\mathrm{eo}} = 0,2 - 0,1i$; $\kappa_{xy}^{\mathrm{e}} = 0,15 + 0,3i$; $\kappa_{xy}^{\mathrm{o}} = 0,15 - 0,1i$; $\kappa_{xy}^{\mathrm{eo}} = 0,1 + 0,1i$; $\kappa_{xz}^{\mathrm{e}} = 0,1 - 0,2i$; $\kappa_{xz}^{\mathrm{o}} = 0,15 + 0,2i$; $\kappa_{xz}^{\mathrm{eo}} = 0,1 - 0,2i$; $\kappa_{yz}^{\mathrm{e}} = 0,15 + 0,2i$; $\kappa_{yz}^{\mathrm{o}} = 0,15 - 0,2i$; $\kappa_{yz}^{\mathrm{eo}} = 0,1 - 0,1i$; $\kappa_{xyz}^{\mathrm{e}} = -0,1 + 0,1i$; $\kappa_{xyz}^{\mathrm{o}} = -0,1 + 0,1i$; $\kappa_{xyz}^{\mathrm{eo}} = 0,1 + 0,1i$

9 Natural oscillations of three-dimensional lattices DR with degenerate modes

The parameters of coupled oscillations of a three-dimensional rectangular lattice of identical $N \times M \times L$ resonators (Fig. 4, a) were calculated for the case when each of them had two degenerate types modes. Keeping the designations of Sections 6 and 7, taking into account the solution for the amplitudes:

$$\mathrm{b}_{s,t,l}^{\mathrm{e,o}} = \mathrm{b}_0^{\mathrm{e,o}} \sin(\theta_x s)\sin(\theta_y t)\sin(\theta_z l), \tag{48}$$

we found a system of equations:

$$\begin{cases} [2\Sigma^{\mathrm{ee}} + (i\tilde{k}_0 - \lambda)]b_0^{\mathrm{e}} + 2\Sigma^{\mathrm{eo}} b_0^{\mathrm{o}} = 0; \\ 2\Sigma^{\mathrm{eo}} b_0^{\mathrm{e}} + [2\Sigma^{\mathrm{oo}} + (i\tilde{k}_0 - \lambda)]b_0^{\mathrm{o}} = 0, \end{cases} \tag{49}$$

where indicated:

$$\begin{aligned} \Sigma^{uv} &= \kappa_x^{uv}\cos(\theta_x) + \kappa_y^{uv}\cos(\theta_y) + \kappa_z^{uv}\cos(\theta_z) + \\ &+ 2\kappa_{xy}^{uv}\cos(\theta_x)\cos(\theta_y) + 2\kappa_{xz}^{uv}\cos(\theta_x)\cos(\theta_z) + \\ &+ 2\kappa_{yz}^{uv}\cos(\theta_y)\cos(\theta_z) + \\ &+ 4\kappa_{xyz}^{uv}\cos(\theta_x)\cos(\theta_y)\cos(\theta_z). \end{aligned} \tag{50}$$

The indices u, v take values ee; oo; eo depending on the type of natural oscillations of the DR.

Non-trivial solutions of the system (49) gave us the relative frequencies of natural oscillations:

$$\begin{aligned} &\lambda^{\pm} = i\tilde{k}_0 + (\Sigma^{\mathrm{ee}} + \Sigma^{\mathrm{oo}}) \pm d, \\ &\text{where} \\ &d = \left\{(\Sigma^{\mathrm{ee}} + \Sigma^{\mathrm{oo}})^2 - 4\left[\Sigma^{\mathrm{ee}}\Sigma^{\mathrm{oo}} - (\Sigma^{\mathrm{eo}})^2\right]\right\}^{1/2} \end{aligned} \tag{51}$$

and

$$b_0^{\mathrm{o}}/b_0^{\mathrm{e}} = -[2\Sigma^{\mathrm{ee}} + (i\tilde{k}_0 - \lambda)]/[2\Sigma^{\mathrm{eo}}]. \tag{52}$$

Fig. 4, c shows the results of comparison of the eigenvalues obtained numerically from (4) for the truncated coupling matrix (dots) and calculated using formula (51) (crosses) for a rectangular lattice $N \times M \times L = 4 \times 4 \times 4$.

Summarizing further the results obtained above, we formulated the following statement:

if the coupling matrix is Toeplitz ($\kappa_{\mathrm{st}} = \kappa_{|\mathrm{s-t}|} = \kappa_n$), and the distribution of the amplitudes of the natural oscillations of identical resonators obeys the relations:

in the case of a **one-dimensional lattice**:

$$\sum_{\mathrm{s}=1}^{\mathrm{N}} \kappa_{\mathrm{st}}\, \mathrm{b}_s = \mathrm{b}_t \sum_{\mathrm{n}=1}^{\mathrm{N}} \kappa_n\, \alpha_n;$$

for **two-dimensional lattice**: ($\kappa_{\mathrm{uv|st}} = \kappa_{|\mathrm{u-s}|,|v-t|} = \kappa_{\mathrm{n,m}}$) and:

$$\sum_{\mathrm{u}=1}^{\mathrm{N}}\sum_{v=1}^{M} \kappa_{\mathrm{uv|st}}\, \mathrm{b}_{uv} = \mathrm{b}_{st} \sum_{\mathrm{n}=1}^{\mathrm{N}}\sum_{m=1}^{M} \kappa_{n,m}\, \alpha_{nm};$$

for **three-dimensional lattice**: ($\kappa_{\mathrm{uvw|str}} = \kappa_{|\mathrm{u-s}|,|v-t|,|w-r|} = \kappa_{n,m,l}$) and:

$$\sum_{\mathrm{u}=1}^{\mathrm{N}}\sum_{v=1}^{M}\sum_{w=1}^{L} \kappa_{\mathrm{uvw|str}}\, \mathrm{b}_{uvw} = \mathrm{b}_{str} \sum_{\mathrm{n}=1}^{\mathrm{N}}\sum_{m=1}^{M}\sum_{l=1}^{L} \kappa_{n,m,l}\, \alpha_{nml},$$

where α_n; α_{nm}; α_{nml} do not depend on t; s, t; s, t, r, respectively, then the frequency spectrum of the lattice with degenerate types of natural oscillations of resonators (Fig. 3) is simply may be expressed through the coupled oscillation characteristics of the same lattice with non-degenerate modes. As follows from the results presented above, the formulated conditions are not sufficient to obtain exact solutions, but they allow the possibility of finding sufficiently accurate eigenvalues in practically the most difficult cases for $N, M, L \gg 1$.

The validity of this proposal is confirmed by the examples given in this paper of the analytical calculation of the parameters of coupled oscillations of rectangular lattices with DR degenerate modes.

Conclusion

It is shown that the presence of symmetry in lattices of identical resonators leads to the appearance of relatively simple real harmonic distributions of the amplitudes of coupled oscillations. Such amplitude distributions do not depend on the electromagnetic parameters of the resonators, but are determined only by the geometric parameters of the lattice.

Analytical expressions for the parameters of natural oscillations of one-, two- and three-dimensional rectangular lattices of identical DRs are found.

The conditions for obtaining the first approximation using a tridiagonal matrix for calculating the parameters of coupled oscillations are established.

The conditions under which the frequency spectrum of a resonator lattice with degenerate types of natural oscillations is expressed through the characteristics of the same lattice with non-degenerate resonator oscillations are formulated.

The obtained analytical solutions to the problem of natural oscillations of rectangular lattices with big DR number are significantly simplify the calculation and analysis of their characteristics: amplitudes and frequencies.

References

[1] F. Andreoli, C. R. Mann, A. A. High, D. E. Chang. (2025). Metalens formed by structured arrays of atomic emitters. *Nanophotonics*, Vol. 14, Iss. 3, pp. 375-395, doi:10.1515/nanoph-2024-0603.

[2] P. Garg, J. D. Fischbach, A. G. Lamprianidis, X. Wang, M. S. Mirmoosa et al. (2025). Inverse-Designed Dispersive Time-Varying Nanostructures. *Advanced Optical Materials*, Wiley, Vol. 13, Iss. 5, 2402444, doi: 10.1002/adom.202402444.

[3] A. V. Prokhorov, M. Y. Gubin, A. V. Shesterikov, A. V. Arsenin, V. S. Volkov, A. B. Evlyukhin. (2025). Tunable Resonant Invisibility of All-Dielectric Metasurfaces on a Stretchable Substrate: Implications for Highly Sensitive Opto-Mechanical Modulators. *ACS Applied Nano Materials*, Vol. 8, Iss. 5, pp. 2319–2327, doi:10.1021/acsanm.4c06453.

[4] S. Bej, N. Tkachenko, R. Fickler, T. Niemi. (2025). Ultrafast Modulation of Guided-Mode Resonance in a Nonlinear Silicon Nitride Grating. *Advanced Optical Materials*, Wiley, Vol.13, Iss. 7, 2402632, doi:10.1002/adom.202402632.

[5] S. A. Schulz, R. F. Oulton, M. Kenney, A. Alu, I. Staude et al. (2024). Roadmap on photonic metasurfaces. *Appl. Phys. Lett.*, Vol. 124, Iss. 26, 260701, doi: 10.1063/5.0204694.

[6] G. W. Bidney, J. M. Duran, G. Ariyawansa, I. Anisimov, J. R. Hendrickson, V. N. Astratov. (2024). Micropyramidal Si Photonics–A Versatile Platform for Detector and Emitter Applications. *Laser Photonics Rev.*, Vol. 19, Iss. 3, 2400922, doi:10.1002/lpor.202400922.

[7] M. Matiushechkina, A. B. Evlyukhin, R. Malureanu, V. A Zenin, T. Yezekyan et al. (2024). Design and Experimental Demonstration of Wavelength-Selective Metamirrors on Sapphire Substrates. *Adv. Photonics Res.*, Vol. 6, Iss. 3, 2400116, doi:10.1002/adpr.202400116.

[8] H. Xu, J. Cheng, S. Guan, F. Li, X. Wang et al. (2024). Terahertz single/dual beam scanning with tunable field of view by cascaded metasurfaces. *APL Photon.*, Vol. 9, Iss. 10, 106108, pp. 106108-1–9, doi: 10.1063/5.0233841.

[9] S.-H. Huang, H.-P. Su, C.-Y. Chen, Y.-C. Lin, Z. Yang et al. (2024). Microcavity-assisted multi-resonant metasurfaces enabling versatile wavefront engineering. *Nature Communications*, Vol. 15, Article number: 9658, doi: 10.1038/s41467-024-54057-9.

[10] C. G. Lee, S. Jeon, S. J. Kim, S. J. Kim. (2023). Near-flat top bandpass filter based on non-local resonance in a dielectric metasurface. *Optics Express*, Vol. 31, No. 3, pp. 4920–4931, doi:10.1364/OE.480757.

[11] M. Kim, N.-R. Park, A. Yu, J. T. Kim, M. Jeon et al. (2023). Multilayer all-polymer metasurface stacked on optical fiber via sequential micro-punching process. *Nanophotonics*, Vol. 12, Iss. 13, pp. 2359–2369. doi:10.1515/nanoph-2022-0762.

[12] Y. Yang, J. Seong, M. Choi, J. Park, G. Kim et al. (2023). Integrated metasurfaces for re-envisioning a near-future disruptive optical platform. *Light: Science & Applications*, Vol. 12, Article number: 152, doi:10.1038/s41377-023-01169-4.

[13] Z. Li, L. Zhou, Z. Liu, M. Panmai, S. Li et al. (2023). Modifying the Quality Factors of the Bound States in the Continuum in a Dielectric Metasurface by Mode Coupling. *ACS Photonics*, Vol. 10, Iss. 1, pp. 206-216, doi:10.1021/acsphotonics.2c01461.

[14] J. Wang, J. Du. (2016). Plasmonic and Dielectric Metasurfaces: Design, Fabrication and Applications. *Appl. Sci.*, Vol. 6, Iss. 9, 239; doi:10.3390/app6090239.

[15] V. M. N. Passaro, C. de Tullio, B. Troia, M. La Notte, G. Giannoccaro, F. De Leonardis. (2012). Recent Advances in Integrated Photonic Sensors. *Sensors*,Vol. 12, Iss. 11, pp. 15558-15598; doi:10.3390/s121115558.

[16] M. Mancinelli, R. Guider, M. Masi, P. Bettotti, M. R. Vanacharla et al. (2011). Optical characterization of a SCISSOR device. *Optics Express*, Vol. 19, No. 14, pp. 13664–13674, doi:10.1364/OE.19.013664.

[17] H. C. Liu, A. Yariv. (2011). Synthesis of high-order bandpass filters based on coupled-resonator optical waveguides (CROWs). *Optics Express*, Vol. 19, No. 18, pp. 17653–17668, doi:10.1364/OE.19.017653.

[18] L. Y. M. Tobing, P. Dumon, R. Baets, M. K. Chin. (2008). Boxlike filter response based on complementary photonic bandgaps in two-dimensional microresonator arrays. *Optics Letters*, Vol. 33, No. 21, pp. 2512–2514, doi:10.1364/OL.33.002512.

[19] O. Schwelb, I. Frigyes. (2005). Parallel-coupled phase-matched multiring optical filters. *Microwave and Optical Technology Letters*, Vol. 44, No. 6, pp. 536–540, doi:10.1002/mop.20689.

[20] V. N. Astratov, S. P. Ashili. (2007). Percolation of light through whispering gallery modes in 3D lattices of coupled microspheres. *Optics express*, Vol. 15, No. 25, pp. 17351–17361, doi:10.1364/OE.15.017351.

[21] J. E. Heebner, R. W. Boyd, Q-Han Park. (2002). SCISSOR solitons and other novel propagation effects in microresonator-modified waveguides. *Opt. Soc. Am. B*, Vol. 19, No. 4, pp. 722–731, doi:10.1364/JOSAB.19.000722.

[22] A. Melloni. (2001). Synthesis of a parallel-coupled ring-resonator filter. *Optics Letters*, Vol. 26, No. 12, pp. 917–919, doi:10.1364/OL.26.000917.

[23] R. Bellman. (2012). *Introduction To Matrix Analysis*. Literary Licensing, LLC, 348 p.

[24] I. M. Gel'fand. (1961). *Lectures on Linear Algebra*. Interscience Publishers, 195 p. ISBN: 0470296011,9780470296011.

[25] Trubin A. A. (2024). *Introduction to the Theory of Dielectric Resonators*. Springer International Publishing Switzerland. Series in Advanced Microelectronics. 363 p. doi:10.1007/978-3-031-65396-4.

Зв'язані коливання прямокутних решіток діелектричних резонаторів (аналітичні рішення)

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Розглядаються спектри частот та розподіли амплітуд полів власних коливань систем зв'язаних діелектричних резонаторів (ДР) однакової форми та розмірів, розташованих в одно-, дво- та тривимірних прямокутних решітках. Нехтуючи зв'язками між не сусідніми резонаторами, знайдені загальні аналітичні рішення лінійної однорідної системи рівнянь, запропонованої раніше для опису власних коливань систем зв'язаних ДР. Запропонований алгоритм рішення систем рівнянь зв'язаних коливань однакових ДР, який дозволяє звести рішення до розрахунку визначників тридіагональної та п'ятидіагональної матриць. Показано, що параметри зв'язаних коливань прямокутних структур визначаються через прості гармонічні розподіли амплітуд та частот, які є характерними для Блохівських хвиль квантових частинок в періодичному потенціалі. За допомогою знайдених загальних аналітичних формул розраховані частоти порівнюються з частотами власних коливань, отриманими чисельними методами. Вперше знайдено загальне аналітичне рішення для розподілу амплітуд та частот прямокутної решітки ДР з двократно виродженими типами власних коливань. Показано, що у випадку нульового зв'язку між виродженими коливаннями різних типів, отримані аналітичні вирази переходять у вирази, які описують коливання ДР в більш простих прямокутних решітках. Сформульовані загальні умови, при виконанні яких рішення рівнянь для зв'язаних коливань систем ДР може бути знайдено в аналітичному вигляді. Показано, що при виконанні вказаних умов розподіл амплітуд зв'язаних коливань однакових решіток ДР з виродженими та не виродженими типами власних коливань пов'язані між собою. В цьому випадку запропонована методика розрахунку амплітуд та частот зв'язаних коливань решіток ДР із виродженими коливаннями. Отримані формули дозволяють у загальному вигляді оцінити характеристики спектра власних коливань при збільшенні числа резонаторів у прямокутних решітках. На декількох прикладах демонструється практичне співпадіння знайдених аналітичних та чисельних результатів. Отримані висновки теорії суттєво спрощують розрахунок і оптимізацію параметрів розсіювання різноманітних пристроїв зв'язку мікрохвильового інфрачервоного та оптичного діапазонів довжин хвиль, які побудовані на основі використання прямокутних структур ДР.

Ключові слова: діелектричний резонатор; зв'язані коливання; прямокутна решітка; спектральна теорія